UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

ELLIPTIC CURVES AND PARAMODULAR FORMS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

MANAMI ROY
Norman, Oklahoma
2019

ELLIPTIC CURVES AND PARAMODULAR FORMS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

Dr. Ralf Schmidt, Chair

Dr. Anne Dunn

Dr. Lucy Lifschitz

Dr. Ameya Pitale

Dr. Alan Roche

Dedicated

to

my parents, my brother

and

all my teachers

Acknowledgements

"I am thankful to everyone who has been a part of my journey of graduate school."

First, I would like to thank Dr. Ralf Schmidt from the bottom of my heart for accepting me as one of his doctoral students, for continuously motivating me, and for going beyond his duties in helping me during each phase of my graduate program. I cannot thank him enough for patiently answering all my questions, for correcting my mistakes, for encouraging me and supporting me in difficult circumstances, and above all for being a great teacher and mentor. In the last five years, I have learned a lot from him; his mentorship would play a significant role in my future academic achievements.

I am grateful to my committee members Dr. Anne Dunn, Dr. Lucy Lifschitz, Dr. Kimball Martin, Dr. Ameya Pitale, and Dr. Alan Roche for their valuable time and their constructive comments on my dissertation. Special thanks to Dr. Cris Poor and Dr. David Yuen for helpful discussions.

It has been a pleasure to have a friendly faculty and staff in the mathematics department at OU. I would especially like to thank Dr. Noel Brady, Dr. Michael Jablonski, Dr. Jonathan Kujawa, Dr. Andy Miller, and Dr. Murad Özaydin for their time and kind support. Special thanks go to Dr. Sepideh Stewart for the discussions during RUME courses, for the advice on writing my teaching statement, and for always being supportive. Thank you Lorraine Inkster and Cristin Yates for patiently helping me throughout these years.

I appreciate the resources available for graduate students in the mathematics department and at OU. I have benefited from the seminars and financial support in the mathematics department. I am thankful for travel support provided by the mathematics department as well as by the College of Arts and Sciences at OU.

My graduate life would have been much harder without my friends and fellow graduate students in the mathematics department. Thank you, Chloe, for being such an awesome friend and officemate, and most importantly for being there during all the ups and downs of my graduate school; I can not imagine going through graduate school without you. Thank you, Mahesh, for being a wonderful colleague and friend, and for always helping me. Special thanks to Shaoyun, Long, Siddhesh, Alok, Colin and Jordan, my fellows in the number theory group, for all the mathematical and non-mathematical conversations we had, and for all their help. I would like to thank all my other colleagues and friends in the department, in particular, Dania, Jieru, Elizabeth, Tony, Thomas Lane, Connor, Jessi, Tia, and Ryan Reynolds.

A special thank you to my master's thesis advisor Dr. Kaneenika Sinha for inspiring me to work in the area of number theory, for encouraging me to apply for graduate school in the US, and for being there for me whenever I needed her advice.

I am very fortunate to have some beautiful friends and a loving host family in Norman who have made my life enjoyable outside graduate school. I am also thankful to my close friends from India for being a source of joy at all times.

Finally, I must thank my family: my parents and my brother. Thank you for giving me mental support consistently, for encouraging me to keep going during difficult times, for loving me unconditionally, and for always believing in me.

Contents

List of Tables

List of Figures

Abstract

There is a lifting from a non-CM elliptic curve $E/\mathbb{Q}$ to a cuspidal paramodular newform f of degree 2 and weight 3 given by the symmetric cube map. We find a description of the level of f in terms of the coefficients of the Weierstrass equation of E. In order to compute the paramodular level, we need a detailed description of the local representations π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ attached to $E/\mathbb{Q}_p$, where $\pi \cong \bigotimes_p \pi_p$ is the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ associated with $E/\mathbb{Q}$. We use the available description of the local representations of $\mathrm{GL}(2, \mathbb{Q}_p)$ attached to E for $p \geq 5$ and determine the local representation of $\mathrm{GL}(2, \mathbb{Q}_3)$ attached to E. In fact, we study the representations of $\mathrm{GL}(2, K)$ attached to E/K for any non-archimedean local field K of characteristic 0 and residue characteristic 3.

Chapter 1

Introduction

1.1 Motivation and the problem

Ramakrishnan and Shahidi proved a lifting from a non-CM elliptic curve E over $\mathbb{Q}$ to a Siegel cusp form f of degree 2 and weight 3 in [35]. Behind this lifting from elliptic curves over $\mathbb{Q}$ to Siegel cusp forms of degree 2, there is a functorial transfer of automorphic representations of GL(2) known as the symmetric cube transfer (or the symmetric cube lifting) proven by Kim and Shahidi in [19]. See Chapter 3 for a description of the symmetric cube lifting given by the symmetric cube map (3.1). Before proving the symmetric cube lifting of automorphic representations from GL(2) to GL(4), Kim and Shahidi have studied the symmetric cube L-functions attached to nonmonomial cuspidal representations of GL(2) over an arbitrary number field in [17] and proved that these symmetric cube L-functions are entire. In order to understand the symmetric cube lifting in the classical setting from elliptic curves to Siegel modular forms, one needs to study the representation theoretic phenomena behind it.

Our goal is to better understand the Siegel cusp forms of degree 2 coming from the symmetric cube lifting using some recent results which were not available at that time. In order to study Siegel modular forms, it is important to specify a congruence subgroup. Some natural questions are: Which congruence subgroup should we consider to study the Siegel modular forms coming from the symmetric cube lifting? What is the level of a Siegel modular form obtained by this transfer with respect to a specific congruence subgroup? We consider holomorphic Siegel modular forms of degree 2 with respect to the paramodular group of level M,

$$K(M) = \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & M^{-1}\mathbb{Z} \\ \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{1.1}$$

We explain why we focus on the paramodular group instead of any other congruence subgroups of $\mathrm{GSp}(4,\mathbb{Q})$ in Section 3.5.

Holomorphic Siegel modular forms of degree 2 with respect to the paramodular group of some level are known as *paramodular forms* of degree 2. The study of paramodular forms has received special attention in the literature in recent years. Paramodular forms are well behaved and interesting objects in many ways; for example, there is a theory of old- and newforms [36], and cuspidal newforms admit a strong multiplicity one theorem [43]. Paramodular forms of weight 2 are also the ones appearing in the famous *paramodular conjecture* formulated in [6]. The paramodular conjecture proposes that every abelian surface A over $\mathbb{Q}$ of conductor M with $\mathrm{End}(A) \cong \mathbb{Z}$ corresponds to a paramodular newform of level M. There is some computational evidence to support this conjecture. For instance, Poor and Yuen have classified Siegel modular cusp forms of weight 2 for the paramodular group $K(p)$ for primes $p < 600$ and noticed that it is consistent with the paramodular conjecture in [34]. There is also a local theory of paramodular fixed vectors, developed in [37], with properties similar to the familiar local newform theory for $\mathrm{GL}(2)$.

At first, we study the vector-valued holomorphic Siegel modular forms of degree 2 coming from holomorphic non-CM newforms on $\mathrm{GL}(2)$ via the symmetric cube transfer. Then we focus on the scalar valued cuspidal paramodular forms of degree 2 coming from non-CM elliptic curves over $\mathbb{Q}$ in order to get a definite formula for the level of the paramodular forms coming via the symmetric cube lifting.

1.2 Main results and strategies

The main results of this thesis can be divided into two categories: "local results" and "global results". By local results we mean local representation theoretic results. In Chapter 4, we study the local representations of $\mathrm{GL}(2, K)$ attached to elliptic curves over a non-archimedean local field K of characteristic zero. Finding a definite characterization of the local representations of $\mathrm{GL}(2, K)$ associated to elliptic curves over K is of independent interest; here we use the description of the local representations of $\mathrm{GL}(2, \mathbb{Q}_p)$ attached to elliptic curves over $\mathbb{Q}_p$ to calculate the level of the paramodular form attached to an elliptic curve over $\mathbb{Q}$. A description of local representations of $\mathrm{GL}(2, K)$ attached to elliptic curves E/K is available in [39] when E has good or potential multiplicative reduction. When elliptic curves E/K have additive but potential good reduction with the residual characteristic of K strictly greater than 3, a similar description can be found in [50]. In this work, we find a description of the local representations of $\mathrm{GL}(2, K)$ attached to elliptic curves E/K in terms of the Weierstrass equations of E when the residual characteristic of K is 3 and E has additive but potential good reduction. In Chapter 5, we investigate the symmetric cube of local representations of $\mathrm{GL}(2, K)$ coming from elliptic curves E/K.

We study the global results related to the automorphic representations of $\mathrm{GSp}(4)$ and the Siegel modular forms of degree 2 coming from the symmetric cube lifting in Chapter 3 and Chapter 6. Note that, attached to every non-CM elliptic curve, there is a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Similarly, Siegel cusp forms (especially cuspidal paramodular forms) of degree 2 are related to cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ (see Section 3.5.1). We first study the symmetric cube transfer of cuspidal automorphic representations from $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ to $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ in Theorem 3.2.1. This theorem is reproduced from Theorem A$'$ in [35], but we give a new proof using some recent techniques. Then, using the connection between non-CM elliptic curves (resp. Siegel cusp forms) and cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ (resp. $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$), we prove results about the paramodular forms coming from non-CM elliptic curves over $\mathbb{Q}$ via the sym^3 lifting in Chapter 6. The following result is one of our main global results from Chapter 6. We write $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ with Γ being the usual gamma function.

Theorem 6.2.2. *Let E be a non-CM elliptic curve over $\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$ and conductor N. Let Δ be the discriminant attached to the given Weierstrass equation and v_p be the p-adic valuation. Let $\Delta' = 3^{-v_3(\Delta)}\Delta$. Suppose that E has good or multiplicative reduction at $p = 2$. Then there is a cuspidal paramodular newform f of degree 2, weight 3 and level M with the following properties:*

(i) The level M of f is given by $M = N \displaystyle\prod_{\substack{p \mid N \\ v_p(\Delta) \not\equiv 0 \bmod 4}} p^2.$

(ii) The completed spin L-function $L(s, f)$ attached to f is given by

$$L(s,f) = \Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s + \frac{1}{2}\right)\prod_{p<\infty} L_p(s,f),$$

where $L_p(s, f) = L_p(s, E, \mathrm{sym}^3)$ for all places p. If $p|N$, then

$$L_p(s,f) = \begin{cases} \frac{1}{1-p^{-3/2-s}} & \text{if } E \text{ has split multiplicative reduction at } p, \\ \frac{1}{1+p^{-3/2-s}} & \text{if } E \text{ has non-split multiplicative reduction at } p, \\ \frac{1}{(1-\alpha p^{-s})(1-\alpha^{-1}p^{-s})} & \text{if } j(E) \in \mathbb{Z}_p \text{ and } v_p(\Delta) \equiv 0 \bmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Here, α is an element of $\mathbb{C}^\times$ such that $|\alpha| = 1$. If the following condition is satisfied

$$j(E) \in \mathbb{Z}_p,\ \ v_p(\Delta) \equiv 0 \bmod 4,\ \ \textit{and}\ \left\{ \begin{array}{ll} (p-1)v_p(\Delta) \not\equiv 0 \bmod 12 & \textit{if}\ p \geq 5, \\ \left(\frac{\Delta'}{3}\right) = -1 & \textit{if}\ p = 3, \end{array} \right\}$$

then $\alpha = i$ (the fourth root of unity).

(iii) The Atkin-Lehner eigenvalues of f at the finite places are given by

$$\eta_p = \begin{cases} -1 & \textit{if}\ p \mid N,\ \textit{and}\ E\ \textit{has split mult. red. at}\ p\ \textit{or satisfies}\ (1.2), \\ w(E/\mathbb{Q}_3) & \textit{if}\ 3 \mid N,\ p = 3\ \textit{and}\ E\ \textit{satisfies}\ \mathrm{S}_6'\ \textit{or}\ \mathrm{S}_6'', \\ 1 & \textit{otherwise}, \end{cases}$$

where S_6', S_6'' are defined in Table 4.1 and the condition (1.2) is given by

$$j(E) \in \mathbb{Z}_p\ \textit{and}\ \left\{ \begin{array}{ll} v_p(\Delta) \equiv 0 \bmod 4, (p-1)v_p(\Delta) \not\equiv 0 \bmod 12 & \textit{if}\ p \geq 5, \\ v_p(\Delta) \equiv 2 \bmod 4, \left(\frac{\Delta'}{3}\right) = -1 & \textit{if}\ p = 3, \end{array} \right\}. \quad (1.2)$$

The quantity $w(E/\mathbb{Q}_3)$ appearing in part (iii) of the theorem is the local root number of $E/\mathbb{Q}_3$; one can compute this quantity using Theorem 1.1 of [24]. The L-function $L(s, f)$ in the theorem satisfies the functional equation $L(s, f) = \varepsilon(s, f)L(1 - s, f)$, where $\varepsilon(s, f) = -\prod_{p<\infty} \eta_p$. For examples, see Tables 6.2 and 6.3.

To calculate the level M of f, and the Atkin-Lehner eigenvalues η_p and the local L-factors of f at the bad primes $p \mid N$, we use the local data like the conductor, the L-factor and the ε-factor of $\mathrm{sym}^3(\pi_p)$ from Chapter 5, where $\pi \cong \otimes_{p\leq\infty}\pi_p$ is the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ attached to $E/\mathbb{Q}$. Note that, $\mathrm{sym}^3(\pi) \cong \otimes_{p\leq\infty}\mathrm{sym}^3(\pi_p)$ is the cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ associated to f. So, this work is mainly based on the local and global representation theory of GL(2) and GSp(4).

Chapter 2

Preliminaries

In this chapter we give a quick introduction to the objects of our study and we also provide a short summary of the key concepts and results that we use later.

2.1 Algebraic groups of study

The reductive algebraic groups we consider in this thesis are the following:

(i) The general linear group

$$\mathrm{GL}(n) := \{g \in M(n \times n) : \det(g) \neq 0\}.$$

The kernel of the determinant map $\det : \mathrm{GL}(n) \to \mathrm{GL}(1)$ is the special linear group $\mathrm{SL}(n)$. We mostly consider $\mathrm{GL}(2)$, $\mathrm{GL}(4)$ and $\mathrm{SL}(2)$ for our purpose.

(ii) The group of symplectic similitudes

$$\mathrm{GSp}(4) := \left\{g \in \mathrm{GL}(4) : {}^tgJg = \lambda(g)J \text{ for } \lambda(g) \in \mathrm{GL}(1)\right\}, \ J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}, \qquad (2.1)$$

where J is a symplectic form given by the above matrix. The function λ is called the multiplier homomorphism. The kernel of this function is the symplectic group $\mathrm{Sp}(4)$ and we have the following exact sequence

$$1 \to \mathrm{Sp}(4) \to \mathrm{GSp}(4) \xrightarrow{\lambda} \mathrm{GL}(1) \to 1.$$

Also, we use the group $\mathrm{PGSp}(4) := \mathrm{GSp}(4)/Z$, where Z is the center of $\mathrm{GSp}(4)$.

Note 2.1.1. *Sometimes we consider* $\mathrm{GSp}(4)$ *with respect to the symplectic form* $J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$. *This version of* $\mathrm{GSp}(4)$ *is called the* classical version *of* $\mathrm{GSp}(4)$. *There is an isomorphism between these two different versions of* $\mathrm{GSp}(4)$ *given by the map which interchanges the first two rows and the first two columns of any matrix.*

(iii) The split orthogonal group

$$\mathrm{SO}(5) := \left\{g \in \mathrm{SL}(5) : {}^t g J g = J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\right\}, \ J = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}. \tag{2.2}$$

It is a well known fact that as algebraic groups $\mathrm{PGSp}(4) \cong \mathrm{SO}(5)$.

2.1.1 Parabolic subgroups of $\mathrm{GSp}(4)$

The group $\mathrm{GSp}(4)$ is one of the main focuses of our study. In order to study induced representations of $\mathrm{GSp}(4)$, it is important to know the shapes of all its parabolic subgroups. There are three different conjugacy classes of parabolic subgroups of $\mathrm{GSp}(4)$: the Borel subgroup B, the Siegel parabolic subgroup P and the Klingen parabolic subgroup Q. The parabolic subgroups of $\mathrm{GSp}(4)$ with the symplectic form $J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$ have the following shapes

$$B = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}, \quad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \quad Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}. \tag{2.3}$$

In $\mathrm{GSp}(4)$ with the symplectic form $J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$, they have the following shapes

$$B = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}, \quad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \quad Q = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix}. \tag{2.4}$$

2.1.2 Congruence subgroups of $\mathrm{GSp}(4, K)$

Let K be a non-archimedean field of characteristic zero with residual characteristic p. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. In this thesis, we consider the vectors in a certain family of representations of $\mathrm{GSp}(4, K)$ fixed by the following compact open subgroups of $\mathrm{GSp}(4, K)$.

(i) The paramodular group of level $\mathfrak{p}^n$

$$K(\mathfrak{p}^n) := \left\{g \in \mathrm{GSp}(4, K) : \det(g) \in \mathfrak{o}_K^\times\right\} \cap \begin{bmatrix} \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K \\ \mathfrak{p}^n & \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o}_K \end{bmatrix}. \tag{2.5}$$

(ii) The Siegel congruence subgroup of level $\mathfrak{p}^n$

$$\Gamma_0(\mathfrak{p}^n) := \mathrm{GSp}(4, \mathfrak{o}_K) \cap \begin{bmatrix} \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K \\ \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K & \mathfrak{o}_K \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o}_K & \mathfrak{o}_K \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o}_K & \mathfrak{o}_K \end{bmatrix}. \tag{2.6}$$

(iii) The principal congruence subgroup of level $\mathfrak{p}^n$

$$\Gamma(\mathfrak{p}^n) := \mathrm{GSp}(4, \mathfrak{o}_K) \cap \begin{bmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1+\mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n & 1+\mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}. \tag{2.7}$$

2.2 Non-archimedean local representations

Let K be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Let ϖ_K be a generator of $\mathfrak{p}$ and $k = \mathfrak{o}_K/\mathfrak{p}$ be the residue field of K of order q. Let ν be the normalized absolute value on K such that $\nu(\varpi_K) = q^{-1}$.

Let G be the group of K-points of an algebraic group defined over K. Then G is a locally compact totally disconnected topological group. All representations from now on will be on a complex vector space. A **representation** (π, V) of G is a complex vector space V along with a homomorphism

$$\pi : G \to \mathrm{Aut}(V), \tag{2.8}$$

where $\mathrm{Aut}(V)$ denotes the invertible $\mathbb{C}$-linear endomorphism of V. We say that V is the representation space or the space of the representation. We denote a representation simply by π or sometimes just by V, whichever is convenient. If $\dim(V) < \infty$, then we say π is a finite dimensional representation. Otherwise, we say π is an infinite dimensional representation. In this text we are only concerned about infinite dimensional representations of G.

Definition 2.2.1. *A representation (π, V) of G is called **smooth** if for every vector $v \in V$, the stabilizer $Stab_G(v) := \{g \in G : \pi(g)v = v\}$ of v in G is open.*

Definition 2.2.2. *A representation (π, V) of G is called **admissible** if it is smooth and for each compact open subgroup K' of G, the space $V^{K'}$ of vectors fixed by K' given by $V^{K'} = \{v \in V : \pi(k')v = v \text{ for all } k' \in K'\}$ is finite dimensional.*

We say a subspace W of V is G-invariant if $\{\pi(g)w : g \in G, w \in W\} \subseteq W$.

Definition 2.2.3. *An admissible representation (π, V) of G is called **irreducible** if the only G-invariant subspaces of V are 0 and V. We say π is reducible if π is not irreducible.*

An *irreducible constituent or irreducible subquotient* of a smooth representation (π, V) of G is an irreducible representation of G isomorphic to W/W' where $W' \subset W$ are G-invariant subspaces of V. A *character* of G is a smooth one-dimensional representation of G, i.e., a continuous homomorphism from G to $\mathbb{C}^\times$.

Definition 2.2.4. *A representation (π, V) of G is called **unitary** if there is a non-degenerate G-invariant Hermitian form on the representation space V.*

Definition 2.2.5. *Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the space of linear functionals on V. Then the representation (π^*, V^*) of G given by*

$$(\pi^*(g)f)(v) = f(\pi(g^{-1})v), \tag{2.9}$$

*where $g \in G$, $f \in V^*$ and $v \in V$ is not necessarily smooth. Let $V^\vee$ be the subspace of V^* consisting all $f \in V^*$ such that the stabilizer $stab_G(f)$ of f under the action of G as in (2.9) is open. Then $(\pi^\vee, V^\vee)$ with the action of G as in (2.9) is a smooth representation of G, called the **contragredient** representation.*

Definition 2.2.6. *For any representation (π, V) of G and a character $\chi : K^\times \to \mathbb{C}^\times$, one can form a **twist** $(\pi \otimes \chi, V)$ of the representation π, where the action of G on V is given by*

$$(\pi \otimes \chi)(g)v = \chi(det(g))\pi(g)v. \tag{2.10}$$

Given two representations (π_1, V_1) and (π_2, V_2) of a group G, a linear map $T : V_1 \to V_2$ satisfying $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$ is called an **intertwining map**. We have the following well-known result about intertwining operators on irreducible admissible representations of G.

Theorem 2.2.7 (Schur's Lemma)**.** *Let (π, V) be an irreducible admissible representation of a totally disconnected locally compact group G and let $T : V \to V$ be an intertwining operator for π. Then there exists a complex number c such that $T(v) = cv$ for all $v \in V$.*

An immediate and useful consequence of Schur's Lemma is that the center Z of G acts by scalars on an irreducible admissible representation (π, V) of G. So, there is a character $\omega_\pi : Z \to \mathbb{C}^\times$, called the **central character** of (π, V), such that

$$\pi(z)(v) = \omega_\pi(z)v \quad \text{for } z \in Z,\ v \in V. \tag{2.11}$$

2.2.1 Induced representations

One of the well-known ways of constructing representations of G is by inducing a representation of a closed subgroup H of G. Let $C_c(G)$ be the complex vector space of all locally constant, compactly supported complex valued functions on G. Suppose G is unimodular, i.e., every left Haar measure of G is also a right Haar measure of G. Let $H = MU$ be a closed subgroup of G such that G/H is compact; here M, U are also closed such that M normalizes U and $M \cap U = 1$. Then U is also unimodular. Now we fix a Haar measure du on U. For $h \in H$ and $u \in U$, we define $\delta_H(h)$ to be the positive number such that for all $f \in C_c(U)$,

$$\int_U f(h^{-1}uh)du = \delta_H(h) \int_U f(u)du.$$

We call $\delta_H(h) : H \to \mathbb{C}^\times$ the modular character of H. Let (σ, W) be a smooth representation of M where $H = MU$. Then the representation $(\mathrm{Ind}_H^G(\sigma), V)$ of G induced from σ is called a **normalized induced representation** of G, where V is given by

$$V := \left\{ f : G \to W | f(mug) = \delta_H(m)^{\frac{1}{2}} \sigma(m) f(g) \text{ for } m \in M, u \in U, g \in G \right\}, \tag{2.12}$$

and the action G on V is defined as

$$\left(\mathrm{Ind}_H^G(\sigma)(g')f\right)(g) = f(gg') \quad \text{for all } g, g' \in G. \tag{2.13}$$

When $H = MU$ is a parabolic subgroup of G where M is the Levi subgroup of H and U is the unipotent radical, we call this induction process a **parabolic induction**.

2.2.2 Representations of $\mathrm{GL}(2, K)$

In this section we discuss the infinite dimensional irreducible admissible representations of $G = \mathrm{GL}(2, K)$. Some general references for this section are [7] and [9]. As parabolic subgroup, we have the standard Borel subgroup B consisting of upper triangular matrices. Let χ_1 and χ_2 be characters of $K^\times$. Let $V(\chi_1, \chi_2)$ be the space of the standard induced representation $\pi(\chi_1, \chi_2)$ of G consisting of all locally constant functions $f : \mathrm{GL}(2, K) \to \mathbb{C}$ with the property

$$f\left(\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] g\right) = |ad^{-1}|^{\frac{1}{2}} \chi_1(a)\chi_2(d) f(g) \text{ for all } g \in G,\ a, d \in K^\times, b \in K. \tag{2.14}$$

The action of G on $V(\chi_1, \chi_2)$ is given by right translations as in (2.13). The induced representation $\pi(\chi_1, \chi_2)$ is admissible, but, not always irreducible, and the central character of $\pi(\chi_1, \chi_2)$ is $\chi_1\chi_2$. It is a well-known fact that $\pi(\chi_1, \chi_2)$ is *irreducible* if and only if $\chi_1\chi_2^{-1} \neq \nu^{\pm 1}$. In that case $\pi(\chi_1, \chi_2)$ is called a **principal series representation** and denoted by $\chi_1 \times \chi_2$.

The induced representation $\pi(\chi\nu^{\frac{1}{2}}, \chi\nu^{-\frac{1}{2}})$ is reducible and has two constituents. The unique irreducible quotient is χ-twist of the trivial representation, which is finite dimensional. The unique irreducible infinite dimensional subrepresentation is called a special representation or a **twisted Steinberg representation**, which we denote by $\chi\mathrm{St}_{\mathrm{GL}(2)}$. Specifically, when $\chi = 1$, i.e., the induced representation is $\pi(\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}})$, the unique irreducible subrepresentation is called the **Steinberg representation** which we denote by $\mathrm{St}_{\mathrm{GL}(2)}$. So, all special representations are obtained as twists of a single one, the Steinberg representation. We have the following result regarding isomorphisms between induced representations.

Theorem 2.2.8. *The irreducible admissible representations $\chi_1 \times \chi_2$ and $\mu_1 \times \mu_2$ of $\mathrm{GL}(2, K)$ are equivalent if and only if χ_1 and χ_2 are equal to μ_1 and μ_2 in some order.*

There is one more kind of infinite dimensional irreducible representations of $\mathrm{GL}(2,K)$, known as the supercuspidal representations. Every irreducible representation that is not a subquotient of some $\pi(\chi_1,\chi_2)$ is called supercuspidal. There is an important class of supercuspidal representations of $\mathrm{GL}(2,K)$ known as **dihedral supercuspidal representations**. We denote a dihedral supercuspidal representation by $\omega_{F,\xi}$, which is associated with a quadratic field extension F/K and a character ξ of $F^\times$ that is not trivial on the kernel of the norm map $N_{F/K}$ from $F^\times$ to $K^\times$. We will give an explicit description of dihedral supercuspidal representations in Section 2.4 in terms of its Langlands parameter. The central character of a dihedral supercuspidal representation $\omega_{F,\xi}$ of $\mathrm{GL}(2,K)$ is $\xi|_{K^\times}\cdot\chi_{F/K}$, where $\chi_{F/K}$ is the quadratic character of $K^\times$ associated to the quadratic extension F/K such that $\chi_{F/K}\left(N_{F/K}(F^\times)\right)=1$. The following is a very useful fact about the dihedral supercuspidal representations of $\mathrm{GL}(2,K)$ with trivial central character. We will use this remark quite often in this article.

Remark 2.2.9. *The central character of the dihedral supercuspidal representation $\omega_{F,\xi}$ of $\mathrm{GL}(2,K)$ is $\xi|_{K^\times}\cdot\chi_{F/K}$, where $\chi_{F/K}$ is the quadratic character of $K^\times$ associated to the quadratic extension F/K such that $\chi_{F/K}\left(N_{F/K}(F^\times)\right)=1$. Here $N_{F/K}$ is the norm map from $F^\times$ to $K^\times$. If $\omega_{F,\xi}$ has trivial central character, i.e., $\xi|_{K^\times}\cdot\chi_{F/K}=1$, then by evaluating ξ at $N_{F/K}(y)$ for any $y\in F^\times$, we get $\xi^\sigma=\xi^{-1}$ on $F^\times$.*

Conductors, L-factors and ε-factors.

There is some important data associated to each type of irreducible admissible representations of $\mathrm{GL}(2,K)$ which are used when we study connections between local representation theory and theory of modular forms. For an integer $n\geq 0$, we define a congruence subgroup $\Gamma_0(\mathfrak{p}^n)$ of $\mathrm{GL}(2,K)$ as follows

$$\Gamma_0(\mathfrak{p}^n)=\left[\begin{smallmatrix}\mathfrak{o}_K & \mathfrak{o}_K\\ \mathfrak{p}^n & \mathfrak{o}_K\end{smallmatrix}\right]\cap \mathrm{GL}(2,\mathfrak{o}_K). \tag{2.15}$$

Definition 2.2.10. *Let (π, V) be an infinite-dimensional irreducible admissible representation of* $\mathrm{GL}(2, K)$*. Let $n \geq 0$ be the minimal integer such that the space $V^{\Gamma_0(\mathfrak{p}^n)}$ of vectors fixed by $\Gamma_0(\mathfrak{p}^n)$ is nonzero. We say the conductor $a(\pi)$ of π is n. If $a(\pi) = 0$, we say π is unramified; otherwise it is ramified.*

An important fact is that the dimension of $V^{\Gamma_0\left(\mathfrak{p}^{a(\pi)}\right)}$ is 1. A vector in $V^{\Gamma_0\left(\mathfrak{p}^{a(\pi)}\right)}$ is called a **new vector** (or a local newform), and is analogous to the notion of newform in the theory of modular forms. For a character χ of $K^\times$, we say that χ has conductor $a(\chi) = n$ if n is the smallest non-negative integer such that $\chi|_{1+\mathfrak{p}^n\mathfrak{o}_K} = 1$. Now we give a list of the conductors $a(\pi)$ for the infinite-dimensional irreducible admissible representations of $\mathrm{GL}(2, K)$.

- For a principal series representation $\pi = \chi_1 \times \chi_2$, $a(\pi) = a(\chi_1) + a(\chi_2)$.
- For a twist of the Steinberg representation $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)}$, $a(\pi) = 2a(\chi)$ if χ is ramified and $a(\pi)$=1 if χ is unramified.
- For a supercuspidal representation π, we know $a(\pi) \geq 2$.

To the infinite-dimensional irreducible admissible representations π of $\mathrm{GL}(2, K)$ one can associate certain functions called L-factors $L(s, \pi)$ and ε-factors $\varepsilon(s, \pi, \psi)$, where ψ is an additive character of K. One can define $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ using the theory of zeta integrals; see Chapter 6 in [14] or Chapter 4 in [7]. For the purpose of this text, we emphasize another way of assigning the conductor, the L-factor and the ε-factor to π using its L-parameter and the local Langlands correspondence. We discuss this in Section 2.5.

2.2.3 Representations of $\mathrm{GSp}(4, K)$

Now we consider the infinite dimensional irreducible admissible representations of $G = \mathrm{GSp}(4, K)$. Some good references for this section are [37] and [40]. There are two classes of irreducible admissible representations of $\mathrm{GSp}(4, K)$. The first class consists of all those representations that can be obtained as subquotients of parabolically induced representations from one of the parabolic subgroups B, P and Q as in (2.3). Sally and Tadić have classified these representations in [40] (also see section 2.2 in [37]). The representations in the first class are also known as *non-supercuspidal representations* of $\mathrm{GSp}(4, K)$. The second class consists of all the other representations; these are called *supercuspidal*. The supercuspidal representations of $\mathrm{GSp}(4, K)$ are not classified as well as in the case of $\mathrm{GL}(2, K)$, even for odd characteristic. Here we give a summary of the non-supercuspidal representations of $\mathrm{GSp}(4, K)$; these are separated into eleven groups as in Table A.1 of [37]. Each group contains irreducible constituents of a parabolically induced representation of $\mathrm{GSp}(4, K)$. If an irreducible non-supercuspidal representation π of $\mathrm{GSp}(4, K)$ is a constituent of the parabolically induced representation of Group #, where "#" is the name of a group in Table A.1 of [37], then we say π belongs to the group # or π is a "Type # representation" or sometimes we just say π is of Type #.

Borel-induced representations. Every element h of the Borel subgroup B can be written in the form

$$h = \begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix}, \quad \text{with } a, b, c \in K^\times. \tag{2.16}$$

Let χ_1, χ_2 and σ be characters of $K^\times$, and consider the character of B given by

$$\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c). \tag{2.17}$$

Then the normalized Borel-induced representation of $\mathrm{GSp}(4, K)$ obtained by the character (2.17) of B is denoted by $\chi_1 \times \chi_2 \rtimes \sigma$ and the representation space consists of all locally constant functions $f : \mathrm{GSp}(4, K) \to \mathbb{C}^\times$ such that

$$f\left(\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} g\right) = |a^2 b||c|^{-\frac{3}{2}} \chi_1(a)\chi_2(b)\sigma(c) f(g) \quad \text{for all } g \in \mathrm{GSp}(4, K). \tag{2.18}$$

The group $\mathrm{GSp}(4, K)$ acts on this space by right translations as defined in (2.13). All the irreducible constituents of Borel-induced representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$ are listed within group I to VI in the Table A.1 of [37]. The central character of $\chi_1 \times \chi_2 \rtimes \sigma$ is $\chi_1\chi_2\sigma^2$.

Siegel-induced representations. Every element h of the Siegel parabolic subgroup P can be written in the form

$$h = \left[\begin{smallmatrix} A & * \\ & \lambda A' \end{smallmatrix}\right], \quad \text{with } A \in \mathrm{GL}(2),\ \lambda \in K^\times \text{ and } A' = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] {}^tA^{-1} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]. \tag{2.19}$$

Let (π, V) be an admissible representation of $\mathrm{GL}(2, K)$ and σ be a character of $K^\times$, and consider the representation of P on V given by

$$\left[\begin{smallmatrix} A & * \\ & \lambda A' \end{smallmatrix}\right] \mapsto \sigma(\lambda)\pi(A). \tag{2.20}$$

Then the normalized Siegel-induced representation of $\mathrm{GSp}(4, K)$ obtained from this representation (2.20) of P is denoted by $\pi \rtimes \sigma$ and the representation space consists of all locally constant functions $f : \mathrm{GSp}(4, K) \to V$ such that

$$f\left(\left[\begin{smallmatrix} A & * \\ & \lambda A' \end{smallmatrix}\right] g\right) = |\det(A)\lambda^{-1}|^{\frac{3}{2}}\sigma(\lambda)\pi(A)f(g) \quad \text{for all } g \in \mathrm{GSp}(4, K). \tag{2.21}$$

Again, the group $\mathrm{GSp}(4, K)$ acts on this space by right translations as defined in (2.13). All the irreducible constituents of Siegel-induced representations of the form $\pi \rtimes \sigma$ are listed within group X and XI in the Table A.1 of [37]. The central character of $\pi \rtimes \sigma$ is $\omega_\pi\sigma^2$, where ω_π is the central character of π.

Klingen-induced representations. Every element h of the Klingen subgroup Q can be written in the form

$$h = \begin{bmatrix} \lambda & * & * \\ & A & * \\ & & \lambda^{-1}\det(A) \end{bmatrix}, \text{ with } A \in \mathrm{GL}(2),\ \lambda \in K^\times. \tag{2.22}$$

Let (π, V) be an admissible representation of $\mathrm{GL}(2, K)$ and χ be a character of $K^\times$, and consider the representation of Q on V given by

$$\begin{bmatrix} \lambda & * & * \\ & A & * \\ & & \lambda^{-1}\det(A) \end{bmatrix} \mapsto \chi(\lambda)\pi(A). \tag{2.23}$$

Then the normalized Klingen-induced representation of $\mathrm{GSp}(4, K)$ obtained from the representation (2.23) of Q is denoted by $\chi \rtimes \pi$ and the representation space consists of all locally constant functions $f : \mathrm{GSp}(4, K) \to V$ such that

$$f\left(\begin{bmatrix} \lambda & * & * \\ & A & * \\ & & \lambda^{-1}\det(A) \end{bmatrix} g\right) = |\lambda^2 \det(A)^{-1}| \chi(\lambda)\pi(A)f(g) \quad \text{for all } g \in \mathrm{GSp}(4, K). \tag{2.24}$$

As before, $\mathrm{GSp}(4, K)$ acts on this space by right translations as defined in (2.13). All the irreducible constituents of Klingen-induced representations of the form $\chi \rtimes \pi$ are listed within group VII to IX in the Table A.1 of [37]. The central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$, where ω_π is the central character of π.

2.3 Archimedean local representations

In this section we review some features of representations of the real reductive group $\mathrm{GSp}(4, \mathbb{R})$ that we use in Section 3.5 when we talk about the connection between Siegel modular forms and automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$. Mainly we focus on a class of lowest weight representations of $\mathrm{Sp}(4, \mathbb{R})$ which appears as the infinite component in the automorphic representation attached to a holomorphic Siegel modular form. Our main references for this section are [21], [42].

2.3.1 Root System of $\mathrm{Sp}(4,\mathbb{R})$

We consider $\mathrm{GSp}(4,\mathbb{R})$ and $\mathrm{Sp}(4,\mathbb{R})$ with the symplectic form $J = \left[\begin{smallmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{smallmatrix}\right]$. The Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$ is given by

$$\mathfrak{g} = \left\{X \in M(4,\mathbb{R}) :\ {}^tXJ + JX = 0\right\}.$$

The standard maximal compact subgroup K of $\mathrm{Sp}(4,\mathbb{R})$ is given by

$$K = \left\{\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in \mathrm{GL}(4,\mathbb{R}) :\ A\,{}^tA + B\,{}^tB = \mathbf{1},\ A\,{}^tB = B\,{}^tA\right\}.$$

We have $K \cong U(2)$ via the isomorphism $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \mapsto A + iB$. The Lie algebra $\mathfrak{k}$ of K is given by

$$\mathfrak{k} = \left\{\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in M(4,\mathbb{R}) :\ A = -{}^tA, B = {}^tB\right\}.$$

This is also the 1-eigenspace of the Cartan involution $\theta X = -{}^tX$. The (-1)-eigenspace is

$$\mathfrak{p} = \left\{\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in M(4,\mathbb{R}) :\ A = {}^tA, B = {}^tB\right\},$$

so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The complexification $\mathfrak{p}^{\mathbb{C}}$ of $\mathfrak{p}$ decomposes as $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_+^{\mathbb{C}} \oplus \mathfrak{p}_-^{\mathbb{C}}$, where

$$\mathfrak{p}_\pm^{\mathbb{C}} = \{\left[\begin{smallmatrix} A & \pm iA \\ \pm iA & -A \end{smallmatrix}\right] \in M(4,\mathbb{R}) :\ A = {}^tA\}.$$

Then we have $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}_\pm^{\mathbb{C}}] \subset \mathfrak{p}_\pm^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}_+^{\mathbb{C}} \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}_-^{\mathbb{C}}$. A basis of $\mathfrak{k}$ is given by

$$\left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{smallmatrix}\right].$$

Let $\mathfrak{h}$ be the Cartan subalgebra spanned by the first two elements. The corresponding analytic subgroup H consists of all elements of the form

$$H = \left\{ \begin{bmatrix} \cos(\theta) & & \sin(\theta) & \\ & \cos(\theta') & & \sin(\theta') \\ -\sin(\theta) & & \cos(\theta) & \\ & -\sin(\theta') & & \cos(\theta') \end{bmatrix} : \theta, \theta' \in \mathbb{R}/2\pi i\mathbb{Z} \right\}. \tag{2.25}$$

We consider the following basis for the complexification $\mathfrak{k}^{\mathbb{C}}$,

$$Z = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Z' = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad N_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{bmatrix}, \quad N_- = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{bmatrix}.$$

Then $[Z, N_\pm] = \pm N_\pm$ and $[Z', N_\pm] = \mp N_\pm$. The Cartan subgroup $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is spanned by Z and Z'. Let $(\mathfrak{h}^{\mathbb{C}})' := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$. We define $\mathfrak{g}_\lambda := \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \lambda(H)X$ for all $\lambda \in (\mathfrak{h}^{\mathbb{C}})'\}$. The set Δ of all roots is defined as $\Delta := \{\lambda \in (\mathfrak{h}^{\mathbb{C}})' : \mathfrak{g}_\lambda \neq 0\}$. We identify an element $\lambda \in \mathfrak{h}^{\mathbb{C}}$ with the pair of complex numbers $(\lambda(Z), \lambda(Z'))$. Moreover, we call $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$ an **analytically integral** element of $(\mathfrak{h}^{\mathbb{C}})'$ if $\lambda(Z), \lambda(Z') \in \mathbb{Z}$. If we restrict λ to $\mathfrak{h}$ then the linear map $(\lambda(Z), \lambda(Z'))$ is the derivative of the character

$$\begin{bmatrix} \cos(\theta) & & \sin(\theta) & \\ & \cos(\theta') & & \sin(\theta') \\ -\sin(\theta) & & \cos(\theta) & \\ & -\sin(\theta') & & \cos(\theta') \end{bmatrix} \mapsto e^{i\lambda(Z)\theta + i\lambda(Z')\theta'} \tag{2.26}$$

of the group H. Let E be the $\mathbb{R}$-subspace of $(\mathfrak{h}^{\mathbb{C}})'$ spanned by the root vectors $\pm(1,-1)$, $\pm(0,2)$, $\pm(2,0)$ and $\pm(1,1)$. Here, $\pm(1,-1)$ are the compact roots and others are non-compact roots. The roots form a root system of type B_2 in E. The space $(i\mathfrak{b})'$ appearing in Theorem 9.20 of [21] is our Euclidean space E. Figure 2.1 shows the roots and the analytically integral elements on the plane. An element $\lambda \in E$ is called **non-singular** if $\langle \lambda, \alpha \rangle \neq 0$ for all roots $\alpha \in \Delta$, i.e., if λ does not lie on a wall in Figure 2.1. Suppose R_λ denotes the reflection in the hyperplane perpendicular to the root vector λ. Let W be the Weyl group of this root system, i.e., $W = \langle R_\lambda : \lambda \in \Delta \rangle$. Let W_K be the **compact Weyl group** generated by the reflection in the hyperplane perpendicular to the root

$(1, -1)$.

2.3.2 Minimal K-type and representations of $\mathrm{GSp}(4, \mathbb{R})$

There is an equivalence class of irreducible representations of $K \cong U(2)$ associated to each analytically integral element $\lambda = (\lambda_1, \lambda_2) \in E$ with $\lambda_1 \geq \lambda_2$, we call it the minimal K-type and denote it by $V_{(\lambda_1,\lambda_2)}$. We refer to (λ_1, λ_2) as the highest weight of $V_{(\lambda_1,\lambda_2)}$, and to any non-zero $v_0 \in V_{(\lambda_1,\lambda_2)}$ with this weight as a highest weight vector, i.e., $N_+ v_0 = 0$.

A representation π of $\mathrm{Sp}(4, \mathbb{R})$ is by definition a Harish-Chandra module, i.e., π is a $(\mathfrak{g}, K)$ module. Let $\mathrm{Sp}(4, \mathbb{R})^{\pm}$ be the subgroup of $\mathrm{GSp}(4, \mathbb{R})$ where $\mu(g) \in \{\pm 1\}$. It contains $\mathrm{Sp}(4, \mathbb{R})$ with index 2. Its standard maximal compact subgroup $K^{\pm} := \{\mathrm{diag}(1, 1, -1, -1)\} \ltimes K$. As explained on pg. 4 of [42], a K-type $V_{(\lambda_1,\lambda_2)}$ of $\mathrm{Sp}(4, \mathbb{R})$ with $\lambda_2 \neq -\lambda_1$ induces to a $K^{\pm}$-type of $\mathrm{Sp}(4, \mathbb{R})^{\pm}$ and the K-type $V_{(\lambda,-\lambda)}$ extends to two different $K^{\pm}$-types $V^{+}_{(\lambda,-\lambda)}$ and $V^{-}_{(\lambda,-\lambda)}$ of $\mathrm{Sp}(4, \mathbb{R})^{\pm}$. A representation of $\mathrm{Sp}(4, \mathbb{R})^{\pm}$ is a $(\mathfrak{g}, K^{\pm})$ module. A representation of $\mathrm{GSp}(4, \mathbb{R})$ is a $(\mathfrak{g}', K^{\pm})$ module which is a natural extension of the representation of $\mathrm{Sp}(4, \mathbb{R})^{\pm}$. Here $\mathfrak{g}' \cong \mathbb{R} \oplus \mathfrak{g}$ is the Lie algebra of $\mathrm{GSp}(4, \mathbb{R})$.

A representation π of $\mathrm{Sp}(4, \mathbb{R})$ or $\mathrm{Sp}(4, \mathbb{R})^{\pm}$ or $\mathrm{GSp}(4, \mathbb{R})$ is **admissible**, if each of its K-types occurs with finite multiplicity. In this case we can write π as $\pi = \bigoplus m_\lambda V_\lambda$, where $\lambda = (\lambda_1, \lambda_2)$ runs over analytically integral elements of E with $\lambda_1 \geq \lambda_2$, and m_λ is the multiplicity with which V_λ occurs in π. If $m_\lambda \neq 0$ and λ is closer to the origin than any other λ' with $m_{\lambda'} \neq 0$, then we call V_λ or simply λ a **minimal K-type**.

Discrete series representations of $\mathrm{Sp}(4, \mathbb{R})$.

Harish-Chandra parameterized the discrete series representations X_λ of $\mathrm{Sp}(4, \mathbb{R})$ by analytically integral and non-singular elements $\lambda \in E$, modulo the action of the compact Weyl group of W_K. For such a **Harish-Chandra parameter** λ of X_λ, the lowest K-type of the corresponding discrete series representation is given by the **Blattner pa-**

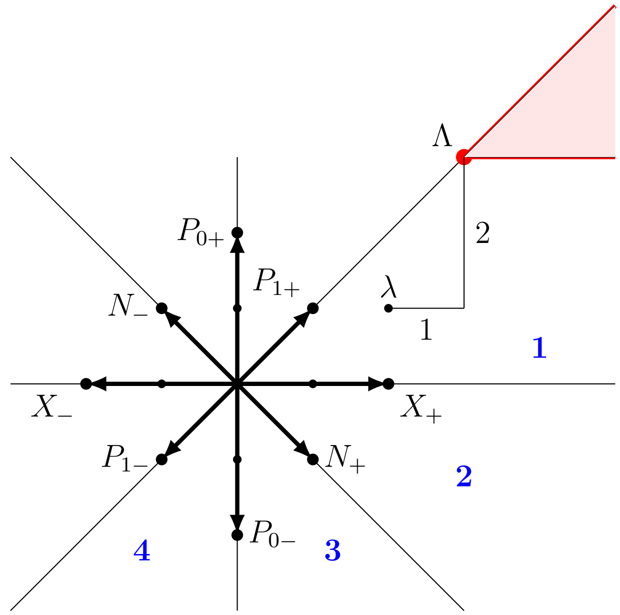

Figure 2.1: The root vectors and the holomorphic discrete series representation with minimal K-type $\Lambda = (3, 3)$.

rameter $\Lambda = \lambda + \delta_{\mathrm{nc}} - \delta_{\mathrm{c}}$, where δ_{nc} (resp. δ_{c}) is half of the sum of the non-compact (resp. compact) positive roots; here "positive" means with respect to the Weyl chamber in which λ lies. The Harish-Chandra parameter λ of a discrete series representation X_λ is in one of the regions 1, 2, 3 or 4 as showed in Figure 2.1.

• If λ is in region 1 (and non-singular, and analytically integral), then we call X_λ a **holomorphic discrete series representation** and it has minimal K-type $\Lambda = \lambda + (1, 2)$. If $\lambda = (k-1, k-2)$ with $k \geq 3$, then $\Lambda = (k, k)$, a one-dimensional K-type. These are the discrete series representations generated by holomorphic Siegel modular forms of weight k. For example, Figure 2.1 (shaded area) shows the holomorphic discrete series representation of minimal K-type $\Lambda = (3, 3)$, which corresponds to a holomorphic Siegel modular form of weight 3 that we encounter in Section 6.

• If λ is in region 2 (and non-singular, and analytically integral), then we call X_λ a **large** (or **generic**) discrete series representation and it has minimal K-type $\Lambda = \lambda + (1, 0)$. If λ is in region 3, then also we have a large discrete series representation.

It is symmetric to the one in region 2 with respect to the diagonal running through the root vectors N_- to N_+ in figure 2.1.

- If λ is in region 4 (and non-singular, and analytically integral), then we call X_λ a **antiholomorphic** discrete series representation. The λ in region 4 is symmetric to the one in region 1 with respect to the diagonal running through the root vectors N_- to N_+ in figure 2.1.

Limits of discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$.

The limits of discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ are parameterized by analytically integral and singular elements $\lambda \in E$ of the forms $(p,0)$, $(0,-p)$ or $(p,-p)$ where $p > 0$ is an integer. When $\lambda = (p,0)$ with $p > 0$, there are two types of representations: a **holomorphic limit** of discrete series representation X^1_λ with minimal K-type $\Lambda = (p+1,2)$ and a **large limit** of discrete series representation X^2_λ with minimal K-type $\Lambda = (p+1,0)$. For $\lambda = (0,-p)$ or $(p,-p)$, we get an **anitiholomorphic limit** of discrete series representation or a **large limit** of discrete series representation.

Non-tempered lowest weight representations of $\mathrm{Sp}(4,\mathbb{R})$.

For $\lambda = (p,1)$ there exists a lowest weight module Y_λ with minimal K-type $\Lambda = (p+1,1)$ (see Proposition 2.8 of [21]). This is a non-tempered representation, we call Y_λ a non-tempered lowest weight representation of $\mathrm{Sp}(4,\mathbb{R})$.

Representations of $\mathrm{GSp}(4,\mathbb{R})$.

Let $\lambda = (\lambda_1,\lambda_2)$ and $\lambda' = (-\lambda_2,-\lambda_1)$ be two analytically integral (singular or non-singular) elements in E. Let π_λ and $\pi_{\lambda'}$ be representations of $\mathrm{Sp}(4,\mathbb{R})$ parametrized by λ and λ' of type either discrete series, or limits of discrete series, or non-tempered lowest weight. Then π_λ and $\pi_{\lambda'}$ are conjugate via $\mathrm{diag}(1,1,-1,-1)$ and we have

$$\pi = \mathrm{ind}_{\mathrm{Sp}(4,\mathbb{R})}^{\mathrm{Sp}(4,\mathbb{R})^\pm}(\pi_\lambda) = \mathrm{ind}_{\mathrm{Sp}(4,\mathbb{R})}^{\mathrm{Sp}(4,\mathbb{R})^\pm}(\pi_{\lambda'}), \tag{2.27}$$

a representation of $\mathrm{Sp}(4,\mathbb{R})^\pm$. The restriction of π to $\mathrm{Sp}(4,\mathbb{R})$ decomposes into a direct sum $\pi_\lambda \oplus \pi_{\lambda'}$, i.e., its K-type is the combination of K-types of π_λ and $\pi_{\lambda'}$. As explained before, one can extend the representation (2.27) to a representation of $\mathrm{GSp}(4,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R})^\pm$ by letting $\mathbb{R}_{>0}$ act trivially.

For example, if λ is a non-singular, analytically integral element of E contained in region 1, then λ' is a non-singular, analytically integral element of E contained in region 4. In this case they generate the same representation of $\mathrm{GSp}(4,\mathbb{R})$ by (2.27), which we call a **holomorphic** discrete series representation X_λ of $\mathrm{GSp}(4,\mathbb{R})$. If λ is a non-singular, analytically integral element in region 2 or 3, then we get a **large or generic** discrete series representation X_λ of $\mathrm{GSp}(4,\mathbb{R})$ by (2.27) .

2.3.3 Lowest weight representations

We say a representation π of $\mathrm{GSp}(4,\mathbb{R})$ is a **lowest weight representation** if it admits a non-zero vector v such that $X_- v = 0$, $P_{1-} v = 0$ and $P_{0-} v = 0$. In this case, the highest weight vector v_0 in any K-type contributing to v is then annihilated by the roots $(-1,1)$, $(0,-2)$, $(-2,0)$ and $(1,-1)$. The possible lowest weight representations are the holomorphic (limits of) discrete series and the non-tempered lowest weight modules.

We define the **weight** of one of these representations to be the pair of non-negative integers (k,j) such that $\Lambda = (k+j,k)$ is the minimal K-type. For each $(k,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$ there exists a unique representation of $\mathrm{GSp}(4,\mathbb{R})$ of type holomorphic discrete series, holomorphic limit of discrete series or the non-tempered lowest weight module with weight (k,j). We denote this unique representation by $\mathcal{B}_{k,j}$. It is a holomorphic discrete series representation if $k \geq 3$, a holomorphic limit of discrete series if $k = 2$, and non-tempered if $k = 1$. We get a uniform description of the L- and ε-factors of these representations if we parametrize them by weight rather than by their Harish-Chandra parameter. The lowest weight representations $\mathcal{B}_{k,j}$ are associated with the vector-valued holomorphic Siegel modular forms of weight (k, j), we will discuss this in Section 3.5.

2.4 Weil-Deligne representations

In this section we recall some basic facts about representations of the Weil group and the Weil-Deligne group. The main references for this section are [49], [39] and [22].

2.4.1 Non-archimedean case

As before, let K be a non-archimedean field of characteristic zero with residual characteristic p. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Let $k = \mathfrak{o}_K/\mathfrak{p}$ be the residue field of K of order q. Let K^{un} be the maximal unramified extension of K inside $\bar{K}$. Let $W(\bar{K}/K)$ be the Weil group of K. It is a subgroup of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$, fitting into the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & I_K \arrow[r] & \mathrm{Gal}(\bar{K}/K) \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1 \\
1 \arrow[r] & I_K \arrow[r] \arrow[u, "="'] & W(\bar{K}/K) \arrow[r] \arrow[u] & \langle \phi \rangle \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Here ϕ is the inverse of the Frobenius automorphism $x \to x^q$ on the residue class field extension $\bar{k}/k$. We fix a pre-image Φ of ϕ in $\mathrm{Gal}(\bar{K}/K)$; such a Φ is called an inverse Frobenius element of $\mathrm{Gal}(\bar{K}/K)$. Then by definition

$$W(\bar{K}/K) = \bigsqcup_{n \in \mathbb{Z}} \Phi^n I_K, \tag{2.28}$$

where I_K is the inertia group, which can be identified with the Galois group $\mathrm{Gal}(\bar{K}/K^{\mathrm{un}})$. The topology on $W(\bar{K}/K)$ is such that I_K is an open subgroup, that the topology on I_K is the Krull topology inherited from $\mathrm{Gal}(\bar{K}/K)$, and that left multiplication by Φ is a homeomorphism. The Weil group comes equipped with the Artin isomorphism $K^\times \cong W(\bar{K}/K)^{\mathrm{ab}}$. Our normalization is such that this isomorphism identifies Φ with a uniformizer ϖ_K of K.

A representation of $W(\bar{K}/K)$ is a continuous homomorphism $\varphi : W(\bar{K}/K) \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space. The continuity condition is equivalent to the requirement that φ be trivial on an open subgroup of I_K. We say that φ is ramified or unramified according as $\varphi|_{I_K}$ is nontrivial or trivial. Let ω be the one-dimensional representation of $W(\bar{K}/K)$ with the property $\omega(I_K) = 1$ and $\omega(\Phi) = q^{-1}$. Then ω corresponds to the normalized absolute value $|\cdot|$ on $K^\times$ via the Artin isomorphism $K^\times \cong W(\bar{K}/K)^{\mathrm{ab}}$.

The Weil-Deligne group $W'(\bar{K}/K)$ is defined as $W'(\bar{K}/K) = W(\bar{K}/K) \ltimes \mathbb{C}$, with the action of $W(\bar{K}/K)$ on $\mathbb{C}$ given by $gzg^{-1} = \omega(g)z$ for $g \in W(\bar{K}/K)$ and $z \in \mathbb{C}$. A representation of $W'(\bar{K}/K)$ is a continuous homomorphism $\varphi' : W'(\bar{K}/K) \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space, such that the restriction of φ' to $\mathbb{C}$ is complex analytic.

More important than to know the formal definition of the Weil-Deligne group is to know the following fact about its representations. There is a one-to-one correspondence between representations φ' of $W'(\bar{K}/K)$ and pairs (φ, N), where φ is a representation of $W(\bar{K}/K)$ and N is a nilpotent endomorphism of V with the property

$$\varphi(g)N\varphi(g)^{-1} = \omega(g)N. \tag{2.29}$$

The relationship is that $\varphi'|_{W(\bar{K}/K)} = \varphi$ and $\varphi'(z) = \exp(zN)$ for $z \in \mathbb{C}$. If φ' and the pair (φ, N) are related in this way, we shall simply write $\varphi' = (\varphi, N)$. We could have defined the notion of "Weil-Deligne representation" as pairs (φ, N) without actually defining the Weil-Deligne group. Since representations of the Weil group are naturally identified with pairs $(\varphi, 0)$, the notion of Weil-Deligne representation is an extension of the notion of Weil group representation.

Here is the basic example for a representation which is honestly Weil-Deligne and not Weil. Let $e_0, ..., e_{n-1}$ be a basis of $\mathbb{C}^n$. Define $\varphi' = (\varphi, N)$ by the following formulas

$$\begin{aligned} &\varphi(g) = \omega(g)^j e_j, \quad \text{for } g \in W(\bar{K}/K), \quad j = 0, ..., (n-1). \\ \text{and} \qquad &Ne_{n-1} = 0, \quad Ne_j = e_{j+1}, \quad j = 0, ..., (n-2). \end{aligned} \tag{2.30}$$

Then (2.29) is satisfied, so that we have indeed defined a Weil-Deligne representation. This representation is denoted by $\mathrm{sp}(n)$ and called the *special representation* of dimension n. In terms of matrices $\varphi' = (\varphi, N)$ has the following form

$$\varphi(g) = \begin{bmatrix} 1 & & & \\ & \omega(g) & & \\ & & \ddots & \\ & & & \omega(g)^{n-1} \end{bmatrix}, \; g \in W(\bar{K}/K), \qquad N = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}. \tag{2.31}$$

Though this is the standard definition of the special representation $\mathrm{sp}(n)$ as in [39], but sometimes for simplicity of calculation we choose an equivalent form of $\mathrm{sp}(n)$ given as follows

$$\varphi(g) = \begin{bmatrix} \omega(g)^{n-1} & & & \\ & \ddots & & \\ & & \omega(g) & \\ & & & 1 \end{bmatrix}, \; g \in W(\bar{K}/K), \qquad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}. \tag{2.32}$$

We call a representation $\varphi' = (\varphi, N)$ of $W'(\bar{K}/K)$ **admissible** if $\varphi(\Phi)$ is a semisimple linear transformation. We say φ' is **indecomposable** if it cannot be written as a direct sum of proper subspaces invariant under $W'(\bar{K}/K)$. We say $\varphi' = (\varphi, N)$ is **irreducible** if φ is irreducible and $N = 0$. It is a well known result that every admissible indecomposable representation of $W'(\bar{K}/K)$ is equivalent to a representation of the form

$\varphi \otimes \mathrm{sp}(n)$, where φ is an irreducible representation of $W(\bar{K}/K)$. Every admissible representation φ' of $W'(\bar{K}/K)$ has a decomposition of the form

$$\varphi' = \bigotimes_{j=1}^{s} \left(\varphi_j \otimes \mathrm{sp}(n_j)\right), \tag{2.33}$$

where φ_j is an irreducible representation of $W(\bar{K}/K)$ and n_j is a positive integer.

Imprimitive representations.

A representation of a group G is called primitive if it is not induced from a proper subgroup. Otherwise it is called imprimitive. We shall describe an important class of imprimitive irreducible representations of $W(\bar{K}/K)$. It turns out that the great majority of irreducible representations of $W(\bar{K}/K)$ are imprimitive; exceptions occur only in residual characteristic 2.

Let F be a quadratic extension of K. Then $W(\bar{K}/F) \subset W(\bar{K}/K)$ is a subgroup of index 2. Let σ be any element of $W(\bar{K}/K)$ that does not lie in $W(\bar{K}/F)$. Then σ induces the non-trivial Galois automorphism of F/K. Let ξ be a character of $W(\bar{K}/F)$. The conjugate character ξ^σ of $W(\bar{K}/F)$ is defined by $\bar{\xi}(x) = \xi(\sigma x \sigma^{-1})$ for $x \in W(\bar{K}/F)$. Assume that ξ is regular, i.e., $\xi \neq \xi^\sigma$. The induced representation

$$\varphi = \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi) \text{ with } \xi \neq \xi^\sigma, \tag{2.34}$$

is irreducible. By Corollary (2.2.5.2) of [49], every irreducible representation of $W(\bar{K}/K)$ is obtained in this way (with some quadratic extension F/K and some regular character ξ of $F^\times$) if the residual characteristic of K is not 2. With respect to a suitable basis, φ has the following matrix form,

$$\varphi(x) = \begin{bmatrix} \xi(x) & \\ & \xi^\sigma(x) \end{bmatrix}, \quad x \in W(\bar{K}/F), \qquad \text{and} \qquad \varphi(\sigma) = \begin{bmatrix} & 1 \\ \xi(\sigma^2) & \end{bmatrix}. \tag{2.35}$$

2.4.2 Archimedean case

The real Weil group $W_{\mathbb{R}}$ is defined as $W_{\mathbb{R}} = \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times}$, where the multiplication on $\mathbb{C}^{\times}$ is standard, and where j is an element satisfying $j^2 = -1$ and $jzj^{-1} = \bar{z}$ (complex conjugation) for $z \in \mathbb{C}^{\times}$. Here we have an exact sequence

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow W_{\mathbb{R}} \longrightarrow \{\pm 1\} \longrightarrow 1 \tag{2.36}$$

where the third map is determined by sending $\mathbb{C}^{\times} \mapsto 1$ and $j \mapsto -1$.

A (semisimple) representation of $W_{\mathbb{R}}$ is a continuous homomorphism $W_{\mathbb{R}} \to \mathrm{GL}(n, \mathbb{C})$ for some n such that its image consists entirely of semisimple elements. Basic facts about $W_{\mathbb{R}}$ are explained in [22]. Every semisimple representation of $W_{\mathbb{R}}$ is fully reducible. An irreducible semisimple representation of $W_{\mathbb{R}}$ is either one- or two-dimensional. The complete list of one-dimensional representations of $W_{\mathbb{R}}$ is as follows:

$$\varphi_{+,t} : re^{i\theta} \mapsto r^{2t}, \quad j \mapsto 1, \tag{2.37}$$

$$\varphi_{-,t} : re^{i\theta} \mapsto r^{2t}, \quad j \mapsto -1, \tag{2.38}$$

where $t \in \mathbb{C}$, and we write a non-zero complex number as $re^{i\theta}$ with $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The two-dimensional representations of $W_{\mathbb{R}}$ are precisely

$$\varphi_{\ell,t} : re^{i\theta} \mapsto \begin{bmatrix} r^{2t}e^{i\ell\theta} & \\ & r^{2t}e^{-i\ell\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} & (-1)^{\ell} \\ 1 & \end{bmatrix}. \tag{2.39}$$

Here ℓ is a positive integer and $t \in \mathbb{C}$. Often, we will only consider the case $t = 0$; in this case, we write $\varphi_{\pm}$ instead of $\varphi_{\pm,0}$ and φ_{ℓ} instead of $\varphi_{\ell,0}$.

2.5 Langlands correspondence and L-parameter

In this section we discuss the local Langlands correspondence for GL(2) and GSp(4). The main references for this section are [4], [49], [27] and [22].

2.5.1 Non-archimedean case

Let K be a non-archimedean field of characteristic zero and G be a split, connected, reductive, linear, algebraic group over K. Some examples of such group are $\mathrm{GL}(n)$, GSp(4). Let $\hat{G}$ be the dual group of G, which is a complex Lie group whose root system is dual to that of G. For $G = \mathrm{GL}(n)$, $\hat{G} = \mathrm{GL}(n, \mathbb{C})$ and for $G = \mathrm{GSp}(4)$, $\hat{G} = \mathrm{GSp}(4, \mathbb{C})$.

Robert P. Langlands conjectured that there exists a bijection between equivalence classes of admissible homomorphisms $\varphi' : W'(\bar{K}/K) \to \hat{G}$ and finite sets $\Pi(\varphi')$ of isomorphism classes of irreducible admissible representations of $G(K)$. This bijection is known the *local Langlands correspondence* for G and satisfies certain desiderata; see [4] and [27]. We say that φ' is the **Langlands parameter** (or L**-parameter**) for each irreducible admissible representation of G in the finite set $\Pi(\varphi')$. Sometimes we also call the set $\Pi(\varphi')$ the L**-packet** of the admissible homomorphism φ' of $W'(\bar{K}/K)$.

The local Langlands correspondence for $\mathrm{GL}(n, K)$ is known; see [15] and [16]. In fact in this case, there is a one-to-one correspondence between the isomorphism classes of irreducible admissible representations π of $\mathrm{GL}(n, K)$ and the n-dimensional admissible representations $\varphi' = (\varphi, N)$ of $W'(\bar{K}/K)$. Here we list the L-parameters (φ, N) for the infinite dimensional irreducible admissible representations π of $\mathrm{GL}(2, K)$. We identify the characters of $W(\bar{K}/K)$ and $K^\times$ via the Artin isomorphism $K^\times \cong W(\bar{K}/K)^{\mathrm{ab}}$.

- When $\pi = \chi_1 \times \chi_2$, where χ_1, χ_2 are characters of $K^\times$ with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, the

L-parameter is given by

$$\varphi(w) = \begin{bmatrix} \chi_1(w) & \\ & \chi_2(w) \end{bmatrix}, w \in W(\bar{K}/K) \text{ and } N = 0. \tag{2.40}$$

- When $\pi = \chi \mathrm{St}_{\mathrm{GL}(2)}$, where χ is a character of $K^\times$, the L-parameter is given by

$$\varphi(w) = \begin{bmatrix} |w|^{\frac{1}{2}}\chi(w) & \\ & |w|^{-\frac{1}{2}}\chi(w) \end{bmatrix}, w \in W(\bar{K}/K) \text{ and } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.41}$$

- When π is a supercuspidal representation, the L-parameter is an irreducible representation of $W(\bar{K}/K)$ with $N = 0$. If π is a dihedral supercuspidal representation, i.e., $\pi = \omega_{F,\xi}$, where F is a quadratic extension of K and a character ξ of $F^\times$, then the L-parameter is $\varphi = \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi)$ with $\xi \neq \xi^\sigma$ as in (2.34), where ξ is the character of $W(\bar{K}/F)$ associated to ξ of $F^\times$ via the Artin isomorphism $F^\times \cong W(\bar{K}/F)^{\mathrm{ab}}$.

Fact 2.5.1. *If the residual characteristic of K is not 2, then every supercuspidal representation π of $\mathrm{GL}(2, K)$ is isomorphic to a dihedral supercuspidal representation $\pi = \omega_{F,\xi}$, where F is a quadratic extension of K and a character ξ of $F^\times$. In this case π has the following L-parameter as in* (2.35)

$$\varphi(x) = \begin{bmatrix} \xi(x) & \\ & \xi^\sigma(x) \end{bmatrix}, \quad x \in W(\bar{K}/F), \qquad \textit{and} \qquad \varphi(\sigma) = \begin{bmatrix} & 1 \\ \xi(\sigma^2) & \end{bmatrix}.$$

The local Langlands correspondence for $\mathrm{GSp}(4, K)$ is also known; see [12]. Let π be an irreducible admissible representation of $\mathrm{GSp}(4, K)$ with the L-parameter $\varphi' : W'(\bar{K}/K) \to \mathrm{GSp}(4, \mathbb{C})$. Since $\mathrm{GSp}(4, \mathbb{C}) \subset \mathrm{GL}(4, \mathbb{C})$, we can consider φ' as an admissible representation of $W'(\bar{K}/K)$. The L-parameters for non-supercuspidal representations of $\mathrm{GSp}(4,K)$ are listed in Section 2.4 of [37] (also see Table A.7 in [37]).

Conductor, L- and ε-factors of L-parameters

Suppose $\varphi' = (\varphi, N)$ is a representation of $W'(\bar{K}/K)$ acting on the space V. Let $\Phi \in W(\bar{K}/K)$ be an inverse Frobenius element, and let $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{un}}) \subset W(\bar{K}/K)$ be the inertia subgroup. Let $V_N = \mathrm{Ker}(N)$ and $V^{I_K} = \{v \in V : \varphi(g)v = v \text{ for all } g \in I_K\}$ and $V_N^I = V^I \cap V_N$. Then the L-factor of φ' is defined by

$$L(s, \varphi') = \det\left(1 - q^{-s}\varphi(\Phi)|_{V_N^I}\right)^{-1}, \tag{2.42}$$

and the ε-factor of φ' is defined by

$$\varepsilon(s, \varphi', \psi) = \varepsilon(s, \varphi, \psi)\det\left(-q^{-s}\varphi(\Phi)|_{V^I/V_N^I}\right), \tag{2.43}$$

where ψ is an additive character of K with conductor (exponent) 0. These are the definitions from p. 60 of [37].

Another important quantity attached to a representation φ' of $W'(\bar{K}/K)$ is a non-negative integer called the conductor, denoted by $a(\varphi')$ and defined by

$$a(\varphi') = a(\varphi) + \dim(V^I) - \dim(V_N^I), \tag{2.44}$$

where $a(\varphi)$ is the conductor of φ defined as in Section 10 of [39]. For general facts on L-factors, ε-factors and conductors of Weil-Deligne representations φ', see [49] and [39].

Note 2.5.2. *Let π be an irreducible admissible representation of $G(K)$, here we consider $G(K)$ to be either $\mathrm{GL}(2,K)$ or $\mathrm{GSp}(4,K)$. Let φ' be the L-parameter of π. So, φ' is an admissible representation of $W'(\bar{K}/K)$ and L-factor, ε-factor and conductor of φ' are defined as in (2.42), (2.43) and (2.44). Then, as a part of the desiderata of the local Langlands correspondence for $G(K)$, we have*

- *The L-factor of π, $L(s, \pi) = L(s, \varphi')$.*

- *The ε-factor of π, $\varepsilon(s, \pi, \psi) = \varepsilon(s, \varphi', \psi)$.*

- *The conductor of π, $a(\pi) = a(\varphi')$.*

In fact, all the representations in the L-packet $\Pi(\varphi')$ have the same conductor, L-factor, and ε-factor.

As we discussed before in Section 2.2, there are other ways to define $a(\pi)$, $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ for an irreducible admissible representation π of $G(K)$. But, since it is easier to calculate these local data for the L-parameter of π, we consider Note 2.5.2 as the definitions for $L(s, \pi)$, $\varepsilon(s, \pi, \psi)$ and $a(\pi)$.

For irreducible admissible representations of $\mathrm{GL}(2, K)$, one can find these local data in Chapter 4 of [7]. Here, we mention the conductor formula of dihedral supercuspidal representations of $\mathrm{GL}(2, K)$ since we use it multiple times in this text.

Conductor of a dihedral supercuspidal representation.

Suppose φ be an imprimitive representations of $W(\bar{K}/K)$ as defined in Section 2.4, i.e., $\varphi = \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi)$ with $\xi \neq \xi^\sigma$, where F/K is a quadratic extension. Then we have the following formula for the conductor $a(\varphi)$ of φ (see VI.2 of [46]),

$$a(\varphi) = \dim(\xi) d(F/K) + f(F/K) a(\xi), \tag{2.45}$$

where $\dim(\xi) = 1$, ξ being a character; $d(F/K)$ is the discriminant of the field extension F/K; $f(F/K)$ is the residue class degree, and $a(\xi)$ is the conductor of ξ.

Let $\pi = \omega_{F,\xi}$ be a dihedral supercuspidal representation of $\mathrm{GL}(2, K)$. Then, as discussed before, the L-parameter of π is an imprimitive representation. Hence, for residual characteristic of K odd, we have

$$a(\omega_{F,\xi}) = \begin{cases} 2a(\xi) & \text{if } F/K \text{ is unramified,} \\ 1 + a(\xi) & \text{if } F/K \text{ is ramified.} \end{cases} \tag{2.46}$$

Here, we consider ξ as a character of $F^\times$ and $a(\xi)$ is the smallest non-negative integer n such that $\xi|_{1+\mathfrak{p}^n \mathfrak{o}_F} = 1$.

For non-supercuspidal irreducible admissible representations of $\mathrm{GSp}(4, K)$, these local data are listed in Tables A.8 and A.9 of [37].

2.5.2 Archimedean case

This case is quite similar to the non-archimedean case. Let G be a split linear reductive group over $\mathbb{R}$. The local Langlands correspondence is a bijection between admissible homomorphisms $\varphi : W_{\mathbb{R}} \to \hat{G}$, where $W_{\mathbb{R}}$ is the real Weil group, and L-packets of irreducible, admissible representations of $G(\mathbb{R})$.

For $G = \mathrm{GL}(2, \mathbb{R})$, the local Langlands correspondence is such that

$$\varphi_{\ell,t} \longleftrightarrow |\det(\cdot)|^t \otimes \mathcal{D}_\ell, \tag{2.47}$$

where $|\det(\cdot)|^t \otimes \mathcal{D}_\ell$ is the irreducible representation of $\mathrm{GL}(2, \mathbb{R})$ with lowest weight $\ell + 1$ and central character determined by $a \mapsto a^{2t}$, $a > 0$. Here $\varphi_{\ell,t}$ is the two dimensional representation of $W_{\mathbb{R}}$ defined in (2.39).

For $G = \mathrm{GSp}(4, \mathbb{R})$, the L-parameters for each irreducible admissible representation of $\mathrm{GSp}(4, \mathbb{R})$ are listed in section 3 of [42]. Here we mention one case that would be useful in Chapter 5. Let $\lambda = (\lambda_1, \lambda_2)$ be a non-singular, analytically integral element in region 1 and $\lambda' = (\lambda_1, -\lambda_2)$ be a non-singular, analytically integral element in region 2 (see figure 2.1). Then the **holomorphic** discrete series representation X_λ and the **large or**

generic discrete series $X_{\lambda'}$ of $\mathrm{GSp}(4,\mathbb{R})$ form a 2-element L-packet and their common L-parameter is the homomorphism $W_{\mathbb{R}} \to \mathrm{GSp}(4,\mathbb{C})$ given by

$$re^{i\theta} \mapsto \begin{bmatrix} e^{i(\lambda_1+\lambda_2)\theta} &&& \\ & e^{i(\lambda_1-\lambda_2)\theta} && \\ && e^{-i(\lambda_1+\lambda_2)\theta} & \\ &&& e^{-i(\lambda_1-\lambda_2)\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} && (-1)^{\lambda_1+\lambda_2} & \\ &&& (-1)^{\lambda_1+\lambda_2} \\ 1 &&& \\ & 1 && \end{bmatrix}. \tag{2.48}$$

The representation X_λ is one of the possible lowest weight representations defined in Section 2.3.3. The lowest weight representation $\pi = \mathcal{B}_{k,j}$ of weight (k,j) of $\mathrm{GSp}(4,\mathbb{R})$ has the spin L-factor and ε-factor given by

$$L(s,\pi) = \Gamma_{\mathbb{C}}\left(s + \frac{2k+j-3}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{j+1}{2}\right) \text{ and } \varepsilon(s,\pi,\psi) = (-1)^{k+j}. \tag{2.49}$$

Here $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ with Γ being the usual gamma function and ψ is the additive character of $\mathbb{R}$ given by $\psi(x) = e^{2\pi i x}$.

2.6 Automorphic representations

In this section we discuss automorphic representations on the adelic group $G(\mathbb{A}_{\mathbb{Q}})$. Our primary references for this section are [5], [23] and [8].

2.6.1 Automorphic forms

Let $\mathbb{A}_{\mathbb{Q}} := \left\{ a = (a_p)_{p\leq\infty} \in \prod\limits_{p\leq\infty} \mathbb{Q}_p : a_p \in \mathbb{Z}_p \text{ for almost all } p \right\}$. We define the topology on $\mathbb{A}_{\mathbb{Q}}$ to be the one generated by the sets $\prod\limits_{p\leq\infty} U_p$, where U_p open is in $\mathbb{Q}_p$ and $U_p = \mathbb{Z}_p$ for almost all p. With this restricted direct product topology, $\mathbb{A}_{\mathbb{Q}}$ becomes a locally compact topological ring. We call $\mathbb{A}_{\mathbb{Q}}$ the **ring of adeles**. Let G be a connected reductive algebraic group over $\mathbb{Q}$. We consider the adelized group

$$G(\mathbb{A}_{\mathbb{Q}}) = \left\{ g = (g_p)_{p\leq\infty} \in \prod_{p\leq\infty} G(\mathbb{Q}_p) : g_p \in G(\mathbb{Z}_p) \text{ for almost all } p \right\}. \tag{2.50}$$

We denote G_∞ to be the archimedean component, and $G(\mathbb{A}_f)$ be the non-archimedean component of $G(\mathbb{A}_\mathbb{Q})$ such that $G(\mathbb{A}_\mathbb{Q}) = G_\infty \times G(\mathbb{A}_f)$. Let $\mathfrak{g}$ be the (real) Lie algebra of G_∞, $\mathfrak{g}^\mathbb{C}$ be the complexification of $\mathfrak{g}$ and K_∞ be a maximal compact subgroup of the Lie group G_∞. Let $U(\mathfrak{g}^\mathbb{C})$ be the universal enveloping algebra of $\mathfrak{g}^\mathbb{C}$ and $Z(\mathfrak{g}^\mathbb{C})$ be the center of $U(\mathfrak{g}^\mathbb{C})$. Let K be the open compact subgroup $G(\prod\limits_{p<\infty} \mathbb{Z}_p)$ of $G(\mathbb{A}_f)$.

We say that $\Phi : G(\mathbb{A}_\mathbb{Q}) = G_\infty \times G(\mathbb{A}_f) \to \mathbb{C}$ is **smooth** if it is continuous and, when viewed as a function of two arguments (x, y) in $G_\infty \times G(\mathbb{A}_f)$ it is smooth in x for each fixed y and is locally constant of compact support in y for each fixed x. Smoothness in the archimedean variable ensures that we can act on Φ with the Lie algebra $\mathfrak{g}$ via right translation, i.e.,

$$(X\Phi)(g) = \frac{d}{dt}\bigg|_{t=0} \Phi\left(g \exp(tX)\right), \quad \text{for } g \in G(\mathbb{A}_\mathbb{Q}),\ X \in \mathfrak{g}. \tag{2.51}$$

A smooth function $\Phi : G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ is called an **automorphic form** if

(i) $\Phi(\gamma g) = \Phi(g)$ for all $g \in G(\mathbb{A}_\mathbb{Q})$ and $\gamma \in G(\mathbb{Q})$.

(ii) $\Phi(gk) = \Phi(g)$ for all $k \in K_1$, where K_1 is an open subgroup of K i.e., Φ is right-invariant under $G(\mathbb{Z}_p)$ for almost all primes p.

(iii) the span of the right translates of Φ by members of K_∞ is finite-dimensional, i.e., the space spanned by all functions $g \to \Phi(gh)$, where $g \in G(\mathbb{A}_\mathbb{Q})$ and $h \in K_\infty$ is finite-dimensional. This condition is known as **K_∞-finiteness**.

(iv) there exists an ideal J of $Z(\mathfrak{g}^\mathbb{C})$ of finite codimension so that the action of J in the G_∞ variable of $G_\infty \times G(\mathbb{A}_f)$ satisfies $J\Phi = 0$. This is known as **$Z(\mathfrak{g}^\mathbb{C})$-finiteness**.

(v) for each $y \in G(\mathbb{A}_f)$, the function $x \mapsto \Phi(xy)$ on G_∞ satisfies a certain slow-growth condition.

One should think of the property (i) as the actual **automorphic property**, and of all the rest as additional regularity conditions. When $G = \mathrm{GL}(2)$, the property (i) of automorphic forms is related to the transformation property (2.57) of classical modular forms.

An automorphic form Φ is called a **cusp form** if Φ satisfies

$$\Phi(zg) = \chi(z)f(g) \quad \text{for all } z \in Z(\mathbb{A}_{\mathbb{Q}}),\ g \in G(\mathbb{A}_{\mathbb{Q}}), \tag{2.52}$$

for some (unitary) character χ of the center $Z(\mathbb{Q})\backslash Z(\mathbb{A}_{\mathbb{Q}})$ and

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_{\mathbb{Q}})} \Phi(ng)dn = 0 \tag{2.53}$$

for the unipotent radical N of every proper parabolic subgroup of G and for all $g \in G(\mathbb{A}_{\mathbb{Q}})$. If Φ is a cusp form, then the condition (v) is equivalent to the condition that $|\Phi| \in \mathrm{L}^2\left(Z(\mathbb{A}_{\mathbb{Q}})G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})\right)$ (see Theorem 7.3 [23]).

Note 2.6.1. *The space $\mathcal{A}$ of automorphic forms is not invariant under right translations by all of $G(\mathbb{A}_{\mathbb{Q}})$. But we still have right translations by the groups K_∞ and $G(\mathbb{A}_f)$, and the action (2.51) of the Lie algebra $\mathfrak{g}$ on $\mathcal{A}$. So, the space $\mathcal{A}$ becomes a $(\mathfrak{g}, K_\infty)$-module and a $G(\mathbb{A}_f)$-module, but, not an actual representation of $G(\mathbb{A}_{\mathbb{Q}})$. We still address $\mathcal{A}$ as a representation of $G(\mathbb{A}_{\mathbb{Q}})$.*

Now we define an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ observing Note 2.6.1.

Definition 2.6.2. *We say that (π, V) is an **automorphic representation** of $G(\mathbb{A}_{\mathbb{Q}})$ if V is an irreducible subquotient of the space of automorphic forms $\mathcal{A}$ on $G(\mathbb{A}_{\mathbb{Q}})$ with the $(\mathfrak{g}, K_\infty)$- and $G(\mathbb{A}_f)$-module structures. An automorphic representation (π, V) of $G(\mathbb{A}_{\mathbb{Q}})$ is called a **cuspidal automorphic representation**, if V is a subspace of the space of cusp forms on $G(\mathbb{A}_{\mathbb{Q}})$.*

There is an important result called the **Tensor Product Theorem** which states that every irreducible and admissible representation π of $G(\mathbb{A}_{\mathbb{Q}})$ can be written as

$$\pi \cong \bigotimes_{p\leq\infty} \pi_p. \tag{2.54}$$

with irreducible, admissible representations π_p of the local groups $G(\mathbb{Q}_p)$. The theorem holds for any irreducible, admissible π and has nothing to do with π being automorphic. A random choice of local representations π_p will pretty much *never* lead to an **automorphic** representation π. In fact, if π as in (2.54) is automorphic, and if we switch out a *single* π_p for another representation, then it is very likely that you have destroyed the automorphic property. So, automorphy is a very special property. Also, The isomorphism (2.54) is abstract. All you know is that it intertwines the $(\mathfrak{g}, K_\infty)$-module and $G(\mathbb{A}_f)$-module structures. The right hand side of (2.54) knows nothing about the specific model of π.

We write an automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$ as $\pi \cong \otimes_{p\leq\infty}\pi_p$ for representations π_p of $G(\mathbb{Q}_p)$. When π is a cuspidal automorphic representation, we call the vectors in the representation space of π cusp forms. Sometimes we call an automorphic representation $\pi \cong \bigotimes\limits_{p\leq\infty} \pi_p$ a **global representation** of G and the π_p's the **local components** of π.

Let $\pi \cong \otimes_{p\leq\infty}\pi_p$ be an irreducible admissible representation of $G(\mathbb{A}_{\mathbb{Q}})$. Let $r : \hat{G} \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism of Lie groups. Then we define the Langlands L-function of π by

$$L(s, \pi, r) := \bigotimes_{p\leq\infty} L(s, \pi_p, r), \tag{2.55}$$

where $L(s, \pi_p, r)$ are the local L-factors of the representations π_p of $G(\mathbb{Q}_p)$. If the local Langlands correspondence is known for $G(\mathbb{Q}_p)$, then we can write $L(s, \pi_p, r) = L(s, \varphi'_p, r) = L(s, r \circ \varphi'_p)$, where φ'_p is the L-parameter of π_p and $L(s, r \circ \varphi'_p)$ is defined as in (2.42).

When $G = \mathrm{GL}(n)$ and r is the standard (identity) map on $\mathrm{GL}(n, \mathbb{C})$, we simply denote $L(s, \pi, r)$ by $L(s, \pi)$. Similarly, when $G = \mathrm{GSp}(4)$ and $r : \mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(4, \mathbb{C})$, we simply write $L(s, \pi, r)$ as $L(s, \pi)$.

2.6.2 Langlands principle of functoriality

The *Langlands principle of functoriality* is a central conjecture in the Langlands program that describes the relationships between automorphic objects living on two different algebraic groups. Let G, H be two (split) reductive algebraic groups defined over $\mathbb{Q}$. Attached to the groups G and H are their dual groups $\hat{G}$ and $\hat{H}$, which are complex reductive Lie groups whose root systems are dual to those of G and H respectively. By the principle of functoriality, every homomorphism of Lie groups $\hat{G} \to \hat{H}$ should give rise to a "lifting" of automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ to automorphic representations of $H(\mathbb{A}_\mathbb{Q})$ such that the L-functions of the automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ and $H(\mathbb{A}_\mathbb{Q})$ are connected. For a detailed description of the principle of functoriality, see Section 9 and 10 in [23]. Here we give a few examples to illustrate the principle of functoriality and we will see another example of functoriality in Section 3.2.

Example 2.6.3. *Let* $G = \mathrm{GL}(2)$ *and* $H = \mathrm{GL}(1)$*. Then* $\hat{G} = \mathrm{GL}(2, \mathbb{C})$ *and* $\hat{H} = \mathbb{C}^\times$*. We have the following homomorphism of Lie groups*

$$det \colon \mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times.$$

Let $\pi \cong \otimes_{p \leq \infty} \pi_p$ *be an automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$*, where* π_p *is a local representation of* $\mathrm{GL}(2, \mathbb{Q}_p)$*. Then we get the following diagram using the local Langlands correspondence for* $\mathrm{GL}(2, \mathbb{Q}_p)$ *and* $\mathbb{Q}_p^\times$ *discussed in Section 2.5.*

$$\pi_p \xrightarrow{LLC} \Big[\varphi_p \colon W'_{\mathbb{Q}_p} \to \mathrm{GL}(2, \mathbb{C})\Big] \xrightarrow{\det} \Big[det \circ \varphi_p \colon W'_{\mathbb{Q}_p} \to \mathbb{C}^\times\Big] \xrightarrow{LLC} \Pi_p. \tag{2.56}$$

Here we denote the Weil-Deligne group $W'(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ defined in Section 2.4 as $W'_{\mathbb{Q}_p}$. Diagram (2.56) gives the characters Π_p of $\mathbb{Q}_p^\times$ for each prime $p \leq \infty$. Then the Langlands principle of functoriality predicts that $\Pi \cong \otimes_{p\leq\infty}\Pi_p$ is an automorphic representation of $\mathbb{A}_{\mathbb{Q}}^\times$. We have the following property of the local Langlands correspondence: if π_p corresponds to φ_p, then the central character ω_{π_p} of π_p corresponds to $det \circ \varphi_p$. Hence, $\Pi = \omega_\pi$, the central character of π and it is known that $\Pi = \omega_\pi$ is automorphic.

Example 2.6.4. *Let $G = \mathrm{GL}(2)$ and $H = \mathrm{GL}(3)$. Then $\hat{G} = \mathrm{GL}(2,\mathbb{C})$ and $\hat{H} = \mathrm{GL}(3,\mathbb{C})$. The symmetric square map given as follows is a homomorphism of Lie groups*

$$\mathrm{sym}^2 : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(3,\mathbb{C})$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}.$$

Let $\pi \cong \otimes_{p\leq\infty}\pi_p$ be an automorphic representation of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$. Then, using the local Langlands correspondence for $\mathrm{GL}(2,\mathbb{Q}_p)$ and $\mathrm{GL}(3,\mathbb{Q}_p)$, the following diagram

$$\pi_p \xrightarrow{LLC} \left[\varphi_p \colon W'_{\mathbb{Q}_p} \to \mathrm{GL}(2,\mathbb{C})\right] \xrightarrow{\mathrm{sym}^2} \left[\mathrm{sym}^2 \circ \varphi_p \colon W'_{\mathbb{Q}_p} \to \mathrm{GL}(3,\mathbb{C})\right] \xrightarrow{LLC} \Pi_p$$

gives representations Π_p of $\mathrm{GL}(3,\mathbb{Q}_p)$. Again, the principle of functoriality predicts $\Pi \cong \otimes_{p\leq\infty}\Pi_p$ is an automorphic representation of $\mathrm{GL}(3,\mathbb{A}_{\mathbb{Q}})$. This is a theorem due to Gelbart and Jacquet (see [13]).

Now, let us discuss the relationship between L-functions of the automorphic representations involved in the principle of functoriality. Let G, H be two (split) reductive algebraic groups defined over $\mathbb{Q}$ and $\rho : \hat{G} \to \hat{H}$ be a homomorphism of Lie groups. Suppose that the principle of functoriality is true for this map ρ, i.e., an automorphic representation $\pi \cong \otimes_{p\leq\infty}\pi_p$ of $G(\mathbb{A}_{\mathbb{Q}})$ lifts to an automorphic representation $\Pi \cong \otimes_{p\leq\infty}\Pi_p$ of $H(\mathbb{A}_{\mathbb{Q}})$ via the diagram

$$\pi_p \xrightarrow{\text{LLC}} \left[\varphi_p \colon W'_{\mathbb{Q}_p} \longrightarrow \hat{G}\right] \xrightarrow{\rho} \left[\rho \circ \varphi_p \colon W'_{\mathbb{Q}_p} \longrightarrow \hat{H}\right] \xrightarrow{\text{LLC}} \Pi_p.$$

Then we have $L(s, \Pi, r) = L(s, \pi, \rho \circ r)$, where $r : \hat{H} \to \mathrm{GL}(n, \mathbb{C})$ is some Lie group homomorphism. By definition (2.55), $L(s, \pi, \rho \circ r) = \bigotimes_{p \leq \infty} L(s, \pi_p, \rho \circ r)$ and $L(s, \Pi, r) = \bigotimes_{p \leq \infty} L(s, \Pi_p, r)$. By the local Langlands correspondence for both the groups, we get

$$L(s, \Pi_p, r) = L(s, r \circ \rho \circ \varphi_p) = L(s, \pi_p, \rho \circ r).$$

We have $L(s, \Pi) = L(s, \pi, \det)$ for Example 2.6.3, and $L(s, \Pi) = L(s, \pi, \mathrm{sym}^2)$ for Example 2.6.4.

2.7 Modular forms and Siegel modular forms

Classical modular forms have a deep connection with elliptic curves over $\mathbb{Q}$, this connection is famous as *the modularity theorem*. The theory of modular forms also has a well-understood connection with automorphic representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. There is a general principle that "every modular form ϕ originates from an automorphic form Φ on $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$". The theory of Siegel modular forms is connected with automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. We discuss the connection between classical Siegel cusp forms and cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ in Section 3.5.1.

In this chapter we review some definitions and useful facts related to modular forms and Siegel modular forms. There are many good references in the literature for the theory of modular forms and Siegel modular forms. For basic facts about classical modular forms and Siegel modular forms we refer to [7], [25], [10], [1] and [20].

2.7.1 Modular forms

Let $\mathcal{H}$ be the complex upper half plane, i.e., $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. For an integer $N > 0$, we define $\Gamma_0(N) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \bmod N\}$, it is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Note that $\Gamma_0(1) \cong \mathrm{SL}(2,\mathbb{Z})$.

Definition 2.7.1. *Let k be an integer. A function $\phi : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight k and level N if it satisfies the following conditions.*

1. *ϕ is holomorphic.*
2. *ϕ satisfies the transformation property*

$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\phi(\tau) \quad \text{for all } \tau \in \mathcal{H} \text{ and } \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N). \tag{2.57}$$

3. *Let $(\phi|_k\gamma_0)(\tau) := (c\tau+d)^{-k}\phi\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\gamma_0 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$ and $\tau \in \mathcal{H}$. Then $\phi|_k\gamma_0$ satisfies some boundedness condition, i.e., for any $\gamma_0 \in \mathrm{SL}(2,\mathbb{Z})$, $\phi|_k\gamma_0$ admits a Fourier expansion of the form*

$$(\phi|_k\gamma_0)(\tau) = \sum_{n=0}^{\infty} c_n e^{\frac{2\pi i n\tau}{N}} = \sum_{n=0}^{\infty} c_n q_N^n, \quad \text{where } q_N = e^{\frac{2\pi i n\tau}{N}}. \tag{2.58}$$

Let $M_k(\Gamma_0(N))$ be the vector space of modular forms of weight k and level N. We call $\phi \in M_k(\Gamma_0(N))$ a **cusp form** if the constant Fourier coefficient $c_0 = 0$ in (2.58). Let $S_k(\Gamma_0(N))$ be the space of cusp forms of weight k and level N.

There exists a natural inner product on $S_k(\Gamma_0(N))$, known as the **Petersson inner product**. If $\phi_1(\tau), \phi_2(\tau) \in S_k(\Gamma_0(N))$, then one can check that $\phi_1(\tau)\overline{\phi_2(\tau)}\mathrm{Im}(\tau)^k$ is invariant under the transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. We define a measure $d\tau$ on the upper half plane $\mathcal{H}$ as $d\tau = \frac{dxdy}{y^2}$, where $\tau = x + iy \in \mathcal{H}$. The volume $V_{\Gamma_0(N)}$ of $\Gamma_0(N)\backslash\mathcal{H}$ is defined by $V_{\Gamma_0(N)} = \int_{\Gamma_0(N)\backslash\mathcal{H}} d\tau$.

Definition 2.7.2. *Let $\phi_1, \phi_2 \in S_k(\Gamma_0(N))$. The Petersson inner product on $S_k(\Gamma_0(N))$ is defined as*

$$\langle \phi_1, \phi_2 \rangle := \frac{1}{V_{\Gamma_0(N)}} \int_{\Gamma_0(N)\backslash\mathcal{H}} \phi_1(\tau)\overline{\phi_2(\tau)}\mathrm{Im}(\tau)^k d\tau. \tag{2.59}$$

There are natural linear transformations acting on spaces of modular forms, known as the Hecke operators. We recall the definition of the Hecke operators on $M_k(\Gamma_0(N))$.

Definition 2.7.3. *Let m be a positive integer and $\phi(\tau) = \sum_{n=0}^{\infty} c_n q^n \in M_k(\Gamma_0(N))$. Then we define the Hecke operator T_m by its action on ϕ as follows*

$$(\phi|T_m)(\tau) := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) \right) q^n. \tag{2.60}$$

The action of the Hecke operator T_m on $S_k(\Gamma_0(N))$ is stable, i.e., if $\phi \in S_k(\Gamma_0(N))$ then $\phi|T_m \in S_k(\Gamma_0(N))$. A modular form $\phi(\tau) = \sum_{n=0}^{\infty} c_n q^n \in M_k(\Gamma_0(N))$ is called a **Hecke eigenform** if it is a common eigenfunction of all the Hecke operators T_m. If $\phi(\tau) = \sum_{n=0}^{\infty} c_n q^n$ is such a Hecke eigenform, one can show that $c_1 \neq 0$ and we say ϕ is **normalized** if $c_1 = 1$.

If $M \mid N$ then it is easy to see that $S_k(\Gamma_0(M)) \subset S_k(\Gamma_0(N))$. This implies some modular forms in $S_k(\Gamma_0(N))$ can come from lower level. In fact, for any $d \mid \frac{N}{M}$, we have the following map

$$\begin{aligned} S_k(\Gamma_0(M)) &\longrightarrow S_k(\Gamma_0(N)) \\ \phi(\tau) &\longmapsto \phi(d\tau). \end{aligned}$$

We denote the space of "oldforms" at level N by $S_k^{\mathrm{old}}(\Gamma_0(N))$, which is the sum of the images of these spaces for proper divisors M of N. The space of newforms at level N is the orthogonal complement with respect to the Petersson inner product and denoted by $S_k^{\mathrm{new}}(\Gamma_0(N))$. It is a well-known result that the space of newforms has a basis consisting of normalized Hecke eigenforms. A cusp form $\phi \in S_k^{\mathrm{new}}(\Gamma_0(N))$ is called a **newform** if it is a normalized Hecke eigenform.

L-functions and functional equation for modular forms.

For a cusp form $\phi(\tau) = \sum_{n=0}^{\infty} c_n q^n \in S_k(\Gamma_0(N))$, the Hecke L-function $L^{\mathrm{ar}}(s, \phi)$ of ϕ (in arithmetic normalization) is defined as

$$L^{\mathrm{ar}}(s, \phi) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} = \prod_p L_p^{\mathrm{ar}}(s, \phi) = \prod_p \frac{1}{1 - c_p p^{-s} + p^{k-1-2s}}. \tag{2.61}$$

This series is convergent for $\mathrm{Re}(s)$ sufficiently large. The Hecke L-function of ϕ in analytic normalization is defined as follows

$$L(s, \phi) := L^{\mathrm{ar}}\left(s + \frac{k}{2} - \frac{1}{2}, \phi\right). \tag{2.62}$$

The L-function $L^{\mathrm{ar}}(s, \phi)$ (resp. $L(s, \phi)$) has a meromorphic continuation to all s, called **the completed L-function**. Let $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ with Γ being the usual gamma function. Then the completed L-function of ϕ is defined as follows

$$\text{In arithmetic nomalization :} \qquad \Lambda^{\mathrm{ar}}(s, \phi) = \Gamma_{\mathbb{C}}(s) L^{\mathrm{ar}}(s, \phi). \tag{2.63}$$

$$\text{In analytic nomalization :} \qquad \Lambda(s, \phi) = \Gamma_{\mathbb{C}}\left(s + \frac{k}{2} - \frac{1}{2}\right) L(s, \phi). \tag{2.64}$$

If ϕ is a cusp form, then $\Lambda^{\mathrm{ar}}(s, \phi)$ (resp. $\Lambda(s, \phi)$) extends to an analytic function of s; if ϕ is not cuspidal then $\Lambda^{\mathrm{ar}}(s, \phi)$ (resp. $\Lambda(s, \phi)$) has simple poles at $s = 0$ and $s = k$ (resp. $s = 1$). Furthermore, it satisfies an equation, known as the *functional equation*, given by

$$\text{In arithmetic nomalization :} \qquad \Lambda^{\mathrm{ar}}(s, \phi) = (-1)^{k/2} \Lambda^{\mathrm{ar}}(k - s, \phi). \tag{2.65}$$

$$\text{In analytic nomalization :} \qquad \Lambda(s, \phi) = (-1)^{k/2} \Lambda(1 - s, \phi). \tag{2.66}$$

We mostly consider the L-function of a cusp form ϕ in analytic nomalization, since it is easier to compare $L(s, \phi)$ with the L-function $L(s, \pi)$ of the cuspidal automorphic representations π of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ associated to ϕ. In fact, $L(s, \phi) = L(s, \pi)$. For a detailed construction of the automorphic representation π of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ associated with a modular form ϕ, see Section 3.6 of [7] or Section 7 of [8].

2.7.2 Vector-valued Siegel modular forms

Following the notations in Section 3 of [42], we define $U_j \cong \mathrm{sym}^j(\mathbb{C}^2)$ to be the space of all complex homogeneous polynomials of total degree j in the two variables S and T. For an integer k, let us define a representation $(\eta_{k,j}, U_j)$ of $\mathrm{GL}(2, \mathbb{C})$ as follows

$$\eta_{k,j}(g)P(S,T) = \det(g)^k P\left((S,T)g\right) \quad \text{for all } g \in \mathrm{GL}(2,\mathbb{C}) \text{ and } P(S,T) \in U_j. \quad (2.67)$$

This is the irreducible representation $\det^k \times \mathrm{sym}^j$ of $\mathrm{GL}(2, \mathbb{C})$. Let $\mathcal{H}_2$ be the Siegel upper half space of degree 2, i.e., $\mathcal{H}_2$ consists of all symmetric complex 2×2 matrices whose imaginary part is positive definite. Let $\mathrm{GSp}(4, \mathbb{R})^+$ be the group consisting of all elements of $\mathrm{GSp}(4, \mathbb{R})$ with positive multiplier. Here we consider the the classical form of $\mathrm{GSp}(4)$ as in Note 2.1.1. There is an action of $\mathrm{GSp}(4, \mathbb{R})^+$ on $\mathcal{H}_2$ given by

$$gZ = (AZ+B)(CZ+D)^{-1} \quad \text{for all } Z \in \mathcal{H}_2 \text{ and } \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}(4,\mathbb{R})^+. \quad (2.68)$$

Let $C^\infty_{k,j}(\mathcal{H}_2) := \{f : \mathcal{H}_2 \to U_j \text{ such that } f \text{ is smooth}\}$. There is an action of $\mathrm{GSp}(4, \mathbb{R})^+$ on $C^\infty_{k,j}(\mathcal{H}_2)$ as follows

$$(f|_{k,j}g)(Z) = \mu(g)^{\frac{k+j}{2}} \eta_{k,j}(CZ+D)^{-1} f(gZ) \quad \text{for } Z \in \mathcal{H}_2,\ g \in \mathrm{GSp}(4,\mathbb{R})^+. \quad (2.69)$$

Here we have chosen the normalization factor $\mu(g)^{\frac{k+j}{2}}$ such that the center of $\mathrm{GSp}(4, \mathbb{R})^+$ acts trivially. Let Γ' be a congruence subgroup of $\mathrm{Sp}(4, \mathbb{Q})$. Then $f : \mathcal{H}_2 \to U_j$ is called a **Siegel modular form** of degree 2 and weight (k, j) with respect to Γ' if f is holomorphic and satisfies the following transformation property

$$f|_{k,j}\gamma = f \quad \text{for all } \gamma \in \Gamma'. \tag{2.70}$$

Let $M_{k,j}(\Gamma')$ be the space of Siegel modular forms of weight (k,j) with respect to the congruence subgroup Γ'. We call $f \in M_{k,j}(\Gamma')$ a **cusp form** if

$$\lim_{\lambda\to\infty} (f|_{k,j}g)\left(\left[\begin{smallmatrix} i\lambda & \\ & \tau \end{smallmatrix}\right]\right) = 0 \quad \text{for all } g \in \mathrm{Sp}(4,\mathbb{Q}), \tau \in \mathcal{H}.$$

Let $S_{k,j}(\Gamma')$ be the space of Siegel cusp forms of degree 2 and weight (k,j) with respect to the congruence subgroup Γ'. We consider the following congruence subgroups of $\mathrm{Sp}(4,\mathbb{Q})$.

1. The principal congruence subgroup of level M:
   $\Gamma(M) = \left[\begin{smallmatrix} 1+M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & 1+M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & 1+M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \end{smallmatrix}\right] \cap \mathrm{Sp}(4,\mathbb{Z})$.
2. The Siegel congruence subgroup of level M:
   $\Gamma_0(M) = \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right] \cap \mathrm{Sp}(4,\mathbb{Z})$.
3. The paramodular group of level M:
   $K(M) = \left[\begin{smallmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & M^{-1}\mathbb{Z} \\ \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right] \cap \mathrm{Sp}(4,\mathbb{Q})$.

We say a Siegel cusp form f of degree 2 and weight (k,j) has **level** M with respect to a congruence subgroup $\Gamma'(M)$ if $f \in S_{k,j}(\Gamma'(M))$. Here we consider $\Gamma'(M) = \Gamma(M)$, $\Gamma_0(M)$, or $K(M)$ which are defined above.
Similar to the case of modular forms, one can define Hecke operators on $S_{k,j}(\Gamma'(M))$. The completed spin L-function for a Hecke eigenform $f \in S_{k,j}(\Gamma'(M))$ and its functional equation are defined in Section 3.3 of [42] (also see Section 4.6 of [3]).

A Siegel modular form with respect to the paramodular group of some level is called a **paramodular form**. There is a newform theory for the space of paramodular Hecke eigenforms of some weight (k, j) and level M, which is similar to the case of modular forms (see [36]).

2.8 Elliptic curves

One of the primary objects of study in this thesis are elliptic curves. In this chapter we discuss some basic facts about elliptic curves over a field K. Our main references for this section are [39], [48] and [47]. The facts in Subsection 2.8.1 are for elliptic curves over any field K, but in all other subsections we assume K to be a non-archimedean local field of characteristic 0.

2.8.1 Weierstrass equations

Roughly speaking, elliptic curves are algebraic curves of genus one having a specified base point. It is a well known fact that such curves can be written as the locus in $\mathbb{P}^2$ of a cubic equation with only one point, the base point, on the line at ∞ (see Proposition 3.1. in [48]). So, an elliptic curve E defined over a field K has an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{2.71}$$

where $a_1, ..., a_6 \in K$. We should also remember that E always has an extra point $O = [0, 1, 0]$ out at infinity. This equation is called a Weierstrass equation. There are some important quantities attached to a Weierstrass equation defined as follows

$$\begin{aligned}
&b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3,\\
&b_6 = a_3^2 + 4a_6, \quad b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,\\
&c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6,\\
&\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 = \frac{c_4^3 - c_6^2}{1728},
\end{aligned} \tag{2.72}$$

$$j(E) = \frac{c_4^3}{\Delta}.$$

Definition 2.8.1. *The quantity Δ is the **discriminant** of the Weierstrass equation. The class of Δ in $K^\times/K^{\times 12}$ is independent of the choice of Weierstrass equation. The quantity $j(E)$ is the j**-invariant** of the elliptic curve, which is an invariant up to isogeny.*

Note 2.8.2. *Not all Weierstrass equations of the form (2.71) represent elliptic curves. If a curve with an equation the form (2.71) has $\Delta = 0$, then that curve is called singular. An elliptic curve is a non-singular curve. So, to be precise, an elliptic curve E has a Weierstrass equation of the form (2.71) such that $\Delta \neq 0$ and it has an extra point $O = [0, 1, 0]$ out at infinity.*

Suppose a curve given by a Weierstrass equation of the form (2.71) is singular and $f(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$. Then there exists a point $P = (x_0, y_0)$ satisfying $f(P) = 0$, called a singular point, such that

$$\frac{\partial f}{\partial x}(P) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(P) = 0. \tag{2.73}$$

Then there are $\alpha, \beta \in \bar{K}$ such that the Taylor series expansion of $f(x, y)$ at P is of the form

$$f(x, y) - f(x_0, y_0) = ((y - y_0) - \alpha(x - x_0))\,((y - y_0) - \beta(x - x_0)) - (x - x_0)^3. \tag{2.74}$$

If $\alpha \neq \beta$, then the point P is called a *node* and there are two tangent lines at P in this case, which are $(y - y_0) - \alpha(x - x_0)$ and $(y - y_0) - \beta(x - x_0)$. If $\alpha = \beta$, then the point P is called a *cusp* and in this case the tangent line at P is $(y - y_0) - \alpha(x - x_0)$. One can easily graph the real locus of a Weierstrass equation. Some examples of elliptic curves are given in Figure 2.2 and some examples of singular curves are given in Figure 2.3.

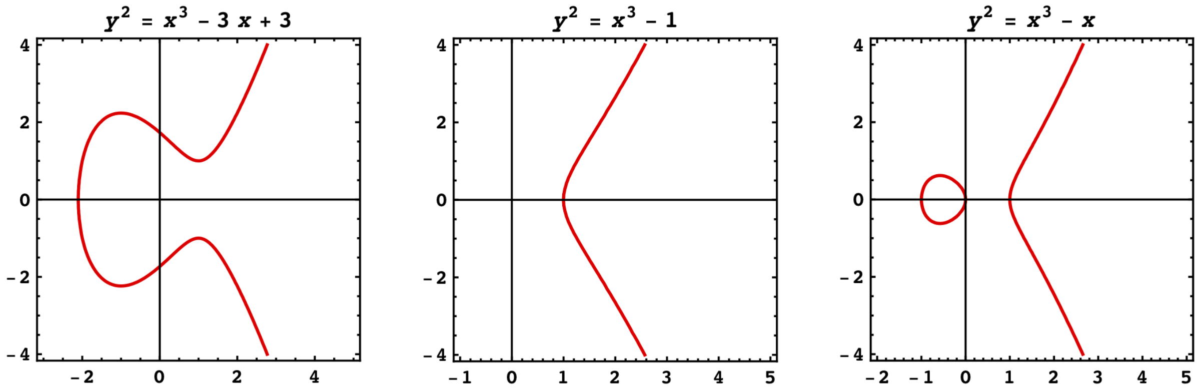

Figure 2.2: Examples of elliptic curves.

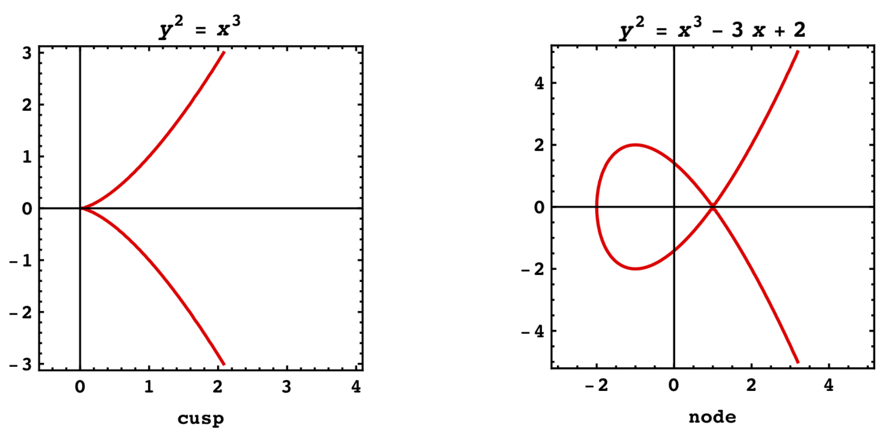

Figure 2.3: Examples of singular curves.

Minimal Weierstrass equations over local fields.

Suppose K is a non-archimedean local field of characteristic 0. Let $\mathfrak{o}_K$ be the ring of integers of K and let $v : K \to \mathbb{Z}$ be the normalized valuation on K. A Weierstrass equation (2.71) of E/K is called **minimal** if $v(\Delta)$ is minimized subject to the condition that all a_i are in $\mathfrak{o}_K$. This minimal value of $v(\Delta)$ is called the *valuation of the minimal discriminant* of E. We have the following facts (see Remark 1.1 in Chapter VII of [48]):

- If all a_i are in $\mathfrak{o}_K$ and $v(\Delta) < 12$, then the equation is minimal.
- If all a_i are in $\mathfrak{o}_K$ and $v(c_4) < 4$, then the equation is minimal.
- If all a_i are in $\mathfrak{o}_K$ and $v(c_6) < 6$, then the equation is minimal.

Every elliptic curve E/K has a minimal Weierstrass equation, and such an equation is unique up to a change of coordinates

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t, \quad \text{where } u \in \mathfrak{o}_K^\times \text{ and } r, s, t \in \mathfrak{o}_K. \tag{2.75}$$

Global minimal Weierstrass equations.

Suppose E is an elliptic curve over $\mathbb{Q}$ with a Weierstrass equation of the form (2.71) and discriminant Δ. One can consider the same Weierstrass equation over $\mathbb{Q}_p$ for each prime p. Let Δ_p be the discriminant of the Weierstrass equation over $\mathbb{Q}_p$. We start with a Weierstrass equation of $E/\mathbb{Q}$ such that the Weierstrass equations over $\mathbb{Q}_p$ are minimal for each prime p. We define the *minimal discriminant* $\mathcal{D}_{E/\mathbb{Q}}$ of $E/\mathbb{Q}$ to be the ideal $\mathcal{D}_{E/\mathbb{Q}} = \prod_p \mathfrak{p}^{v_p(\Delta_p)}$, where $\mathfrak{p}$ is the maximal ideal of $\mathbb{Z}_p$ and v_p is the p-adic valuation.

Definition 2.8.3. *A global minimal Weierstrass equation for $E/\mathbb{Q}$ is a Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ such that a_i are in $\mathbb{Z}$ and such that the discriminant Δ of the equation satisfies $\mathcal{D}_{E/\mathbb{Q}} = (\Delta)$.*

It is a well-known fact that every elliptic curve E over $\mathbb{Q}$ has a global minimal Weierstrass equation (see Corollary 8.3 in [48]). Note that, if $E/\mathbb{Q}$ is given by a global minimal Weierstrass equation with discriminant Δ, then $v_p(\Delta)$ is the valuation of the minimal discriminant of $E/\mathbb{Q}_p$.

Isogenies and CM curves.

Let E_1 and E_2 be two elliptic curves. An *isogeny* ϕ from E_1 to E_2 is a morphism (of algebraic curves) $\phi : E_1 \to E_2$ such that $\phi(O) = O$. For an elliptic curve E, let $\mathrm{End}(E) = \{\phi : E \to E \text{ such that } \phi \text{ is an isogeny}\}$. Then $\mathrm{End}(E)$ is a ring, it is called the *endomorphism ring* of E. Suppose that $\mathrm{char}(K) = 0$. Then the ring homomorphism

$$\begin{aligned} [\ ] : \mathbb{Z} &\longrightarrow \mathrm{End}(E) \\ m &\longmapsto [m] \end{aligned}$$

is usually an isomorphism, i.e., $\mathrm{End}(E) \cong \mathbb{Z}$. Here, $[m] : E \to E$ is the multiplication-by-m map and O is the point at infinity. If $\mathrm{End}(E)$ is strictly larger than $\mathbb{Z}$, then we say that E has **complex multiplication**, or CM for short. If E is an elliptic curve such that $\mathrm{End}(E) \cong \mathbb{Z}$ then we call E a **non-CM elliptic curve**.

2.8.2 Reduction types

Let K be a non-archimedean local field of characteristic zero with residual characteristic p. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Suppose that $v : K \to \mathbb{Z}$ is the normalized valuation on K, and $k = \mathfrak{o}_K/\mathfrak{p}$ is the residue field of K of order q. Let E/K be an elliptic curve. Assume that (2.71) is a minimal Weierstrass equation for E/K. Consider the curve $\tilde{E}/k$ given by

$$y^2 + \tilde{a}_1xy + \tilde{a}_3y = x^3 + \tilde{a}_2x^2 + \tilde{a}_4x + \tilde{a}_6, \tag{2.76}$$

where $\tilde{a}_i$ denotes the image of a_i in k. The equation (2.76) for $\tilde{E}$ depends on the choice of minimal Weierstrass equation for E, but (2.75) shows that any two such choices lead to equations for $\tilde{E}$ that are related by a standard change of coordinates over k. The equivalence class of the curve $\tilde{E}$ is called the reduction of E modulo $\mathfrak{p}$.

Definition 2.8.4. *Let E/K be an elliptic curve, and let $\tilde{E}/k$ be its reduction modulo $\mathfrak{p}$.*

1. *If $\tilde{E}$ is non-singular (i.e., an elliptic curve over k), then E is said to have **good reduction** (or stable reduction).*

2. *If $\tilde{E}$ has a node, then E is said to have **multiplicative reduction** (or semi-stable reduction). In this case, we have two different possibilities:*

   - *If the slopes of the tangent lines at the node are in k, then E is said to have **split multiplicative reduction**.*

   - *If the slopes of the tangent lines at the node are not in k, then E is said to have **non-split multiplicative reduction**.*

3. *If $\tilde{E}$ has a cusp, then E is said to have **additive reduction** (or unstable reduction).*

*In cases 1 and 2, E is said to have **bad reduction**.*

Given the minimal Weierstrass equation of E/K, we have the following standard results to read off the reduction type of E from its minimal Weierstrass equation (see Proposition 5.1 in Chapter VII of [48]):

$$E \text{ has good reduction} \quad \text{if and only if} \quad v(\Delta) = 0. \tag{2.77}$$

$$E \text{ has multiplicative reduction} \quad \text{if and only if} \quad v(\Delta) > 0 \text{ and } v(c_4) = 0. \tag{2.78}$$

$$E \text{ has additive reduction} \quad \text{if and only if} \quad v(\Delta) > 0 \text{ and } v(c_4) > 0. \tag{2.79}$$

In the case of bad reduction, let $\tilde{E}_{\mathrm{ns}}$ be the open subvariety consisting of the non-singular points of $\tilde{E}$. The reduction type can then also be characterized by the cardinality of $\tilde{E}_{\mathrm{ns}}(k)$. Namely

$$E \text{ has split multiplicative reduction } \text{ if and only if } \#\tilde{E}_{\mathrm{ns}}(k) = q - 1. \tag{2.80}$$

$$E \text{ has non-split multiplicative reduction } \text{ if and only if } \#\tilde{E}_{\mathrm{ns}}(k) = q + 1. \tag{2.81}$$

$$E \text{ has additive reduction } \text{ if and only if } \#\tilde{E}_{\mathrm{ns}}(k) = q. \tag{2.82}$$

Good reduction can also be characterized via the action of inertia. Let $\bar{K}$ be an algebraic closure of K and Let $\mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group of K. Let K^{un} be the maximal unramified extension of K inside $\bar{K}$ and $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{un}})$ be the inertia group. There is an exact sequence

$$1 \longrightarrow I_K \longrightarrow \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1. \tag{2.83}$$

Hence, I_K consists precisely of those elements of $\mathrm{Gal}(\bar{K}/K)$ which act trivially on $\bar{k}$. Then we have the following criterion for good reduction.

Theorem 2.8.5 (Criterion of Néron-Ogg-Shafarevich)**.** *Let E/K be an elliptic curve. Let ℓ be a prime different from the characteristic of k. The following statements are equivalent.*

1. *E has good reduction.*
2. *The Tate module $T_\ell(E)$ is unramified, i.e., the inertia group I_K acts trivially on $T_\ell(E)$.*

Reduction type after base extension.

When an elliptic curve E/K has bad reduction, it is often useful to know whether it attains good reduction over some extension of K. Here we list the reduction types of E after its base change to a finite field extension of K. Given E/K and a finite field extension K'/K, we may consider E an elliptic curve over K'. A given minimal Weierstrass equation over K may no longer be a minimal equation over K', i.e., the reduction type may change when extending the field of definition. The following theorem is proved in VII.5 of [48].

Theorem 2.8.6 (Semistable Reduction Theorem)**.** *Let E/K be an elliptic curve.*

1. *Let K'/K be a finite field extension. If E has either good or multiplicative reduction over K, then it has the same reduction type over K'.*

2. *There exists a finite extension K'/K such that E has either good or multiplicative reduction over K'.*

Definition 2.8.7. *Let E/K be an elliptic curve.*

(1) If there exists a finite extension K'/K such that E has good reduction over K', then E is said to have ***potential good reduction****.*

(2) If there exists a finite extension K'/K such that E has multiplicative reduction over K', then E is said to have ***potential multiplicative reduction****.*

By (1) of Theorem 2.8.6, each elliptic curve over K has either potential good or potential multiplicative reduction. Moreover, a given elliptic curve E/K cannot have both potential good and potential multiplicative reduction. Since if E would have good reduction over K_1 and multiplicative reduction over K_2, then E would have both good and multiplicative reduction over K_1K_2, by (2) of Theorem 2.8.6.

Theorem 2.8.8. *Let E/K be an elliptic curve. Let ℓ be a prime different from the characteristic of k. The following statements are equivalent.*

(1) E/K has potential good reduction.

(2) The j-invariant $j(E)$ of E lies in $\mathfrak{o}_K$.

(3) The inertia group I_K acts on $T_\ell(E)$ via a finite quotient (i.e., the image of $I_K \to \mathrm{GL}(2, \mathbb{Q}_\ell)$ is finite).

One can also read off the potential good or potential multiplicative reduction type of E from its minimal Weierstrass equation using Theorem 2.8.8 (see Proposition VII.5.5 and Corollary VII.7.3 of [48]), i.e., we have the following

$$E \text{ has potential multiplicative reduction} \quad \text{if and only if} \quad j(E) \notin \mathfrak{o}_K. \tag{2.84}$$

$$E \text{ has potential good reduction} \quad \text{if and only if} \quad j(E) \in \mathfrak{o}_K. \tag{2.85}$$

2.8.3 Relationship between Weil-Deligne and ℓ-adic representations

We continue assuming that K is a non-archimedean local field of characteristic zero with residual characteristic p and I_K is the inertia group as defined in (2.83). Let k be the residue field of K of order q. In this section we discuss how one can associate a Weil-Deligne representation to an ℓ-adic representation. This association plays an important part to find the Weil-Deligne representation of $W'(\bar{K}/K)$ attached to a given elliptic curve over K. The main references for this section are [39] and [49].

Let ℓ be a prime different from the residual characteristic of K. An ℓ-adic representation of $\mathrm{Gal}(\bar{K}/K)$ is a continuous homomorphism

$$\sigma'_\ell : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}(V_\ell), \tag{2.86}$$

where V_ℓ is a finite-dimensional $\mathbb{Q}_\ell$ vector space. There is a procedure to attach an ℓ-adic representation of $\mathrm{Gal}(\bar{K}/K)$ to a complex representation of $W'(\bar{K}/K)$. This construction will depend on the choice of a field embedding $\iota : \mathbb{Q}_\ell \to \mathbb{C}$ but the isomorphism class of the result will be independent of this choice. Hence, we fix such an embedding. Let σ'_ℓ be a given ℓ-adic representation, as in (2.86). The space of the associated Weil-Deligne representation $\sigma'_{\ell,\iota}$ will be the complex vector space $\mathbb{C} \otimes_\iota V_\ell$. There are two cases to distinguish.

Case I: Assume that the image of the inertia group I_K under the map (2.86) is finite. Equivalently, σ'_ℓ is trivial on an open subgroup J of I_K (which automatically is of finite index). In this case we may simply compose the map (2.86) with the map $\mathrm{GL}(V_\ell) \to \mathrm{GL}\left(\mathbb{C} \otimes_\iota V_\ell\right)$, and restrict the result to $W(\bar{K}/K)$. The resulting representation σ of the Weil group is really continuous since it is trivial on J. In this case we can set $\sigma'_{\ell,\iota} = (\sigma, 0)$. Hence, in this case, $\sigma'_{\ell,\iota}$ is actually a representation of the Weil group.

Case II: Assume that the image of the inertia group I_K under the map (2.86) is infinite. In this case one can prove that there is a canonical way to associate to σ'_ℓ a pair (σ_ℓ, N_ℓ) consisting of a non-zero, nilpotent endomorphism N_ℓ of V_ℓ and a homomorphism $\sigma_\ell : W(\bar{K}/K) \to \mathrm{GL}(V_\ell)$ with the following properties:

- $\sigma_\ell(g) N_\ell \sigma_\ell(g)^{-1} = \omega(g) N_\ell$ for all $g \in W(\bar{K}/K)$. Recall from Section 2.4 that ω is the one-dimensional representation of $W(\bar{K}/K)$ with the property $\omega(I_K) = 1$ and $\omega(\Phi) = q^{-1}$, where Φ is an inverse Frobenius element in $W(\bar{K}/K)$. Note that $\omega(g) N_\ell$ makes sense, since $\omega(g)$ is always a rational number.

- $\sigma_\ell(I_K)$ is finite. Equivalently, σ_ℓ is trivial on an open subgroup J of I_K.

With this σ_ℓ and N_ℓ available, one can proceed as in case I. Composing σ_ℓ with the map $\mathrm{GL}(V_\ell) \to \mathrm{GL}\left(\mathbb{C} \otimes_\iota V_\ell\right)$ gives a complex and continuous representation $\sigma_{\ell,\iota}$. And N_ℓ can be lifted to a nilpotent endomorphism $N_{\ell,\iota}$ of $\mathbb{C} \otimes_\iota V_\ell$ via the map $\mathrm{End}(V_\ell) \to \mathrm{End}\left(\mathbb{C} \otimes_\iota V_\ell\right)$. Then the pair $(\sigma_{\ell,\iota}, N_{\ell,\iota})$ satisfies the compatibility relation (2.29), and hence defines a Weil-Deligne representation $\sigma'_{\ell,\iota}$.

Remark 2.8.9. *The construction of the pair (σ_ℓ, N_ℓ) is done as follows. First, one proves that there is a uniquely determined nilpotent endomorphism N_ℓ of V_ℓ with the* *property that*

$$\sigma'_\ell(i) = exp(t_\ell(i)N_\ell) \tag{2.87}$$

for all i in an open subgroup J of I_K. Here, $t_\ell : I_K \to \mathbb{Q}_\ell$ is a chosen non-trivial continuous homomorphism (unique up to mutiplication by an element of $\mathbb{Q}_\ell^\times$). Then define

$$\sigma_\ell(g) = \sigma'_\ell(g)exp(-t_\ell(i)N_\ell), \quad g = \Phi^m i,\ m \in \mathbb{Z} \text{ and } i \in I_K, \tag{2.88}$$

where Φ is a chosen Frobenius element. Then $\sigma_\ell(I_K)$ is finite, and one can show that the relation $\sigma_\ell(g)N_\ell\sigma_\ell(g)^{-1} = \omega(g)N_\ell$ is automatically satisfied for all $g \in W(\bar{K}/K)$.

For a proof of the above remark, see (4.2.2) of [49] or (the student supplement to) [39].

2.8.4 The Weil-Deligne representation associated to an elliptic curve

As before, let K be a non-archimedean local field of characteristic zero with residual characteristic p and k be the residue field of K of order q. In this section, we describe the process of finding the Weil-Deligne representation associated to an elliptic curve E/K and we give a description the Weil-Deligne representation attached to E/K when E has good reduction. In chapter 4, when the residual characteristic of K is odd, we completely determine the Weil-Deligne representations associated to elliptic curves over K with bad reduction. The main references for this section are [39] and [48].

Definition 2.8.10. *Let E/K be an elliptic curve and let $m \in \mathbb{Z}$ with $m \geq 1$. The m-**torsion subgroup** of E, denoted by $E[m]$, is the set of points of E of order m, i.e,*

$$E[m] := \{P \in E : [m]P = O\}, \tag{2.89}$$

where $[m] : E \to E$ is the multiplication-by-m map and O is the point at infinity.

It is a well-known fact that $E[m] \cong \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}$.

Definition 2.8.11. *Let E/K be an elliptic curve and ℓ be a prime. The* ***(ℓ-adic) Tate module*** *of E is the group*

$$T_\ell(E) := \varprojlim_n E[\ell^n], \tag{2.90}$$

where the inverse limit being taken with respect to the natural maps $E[\ell^{n+1}] \xrightarrow{[\ell]} E[\ell^n]$.

Now, let us assume that ℓ is a prime different from the residual characteristic of K. Then $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2, i.e., $T_\ell(E) \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$. There is a natural action of $\mathrm{Gal}(\bar{K}/K)$ on $T_\ell(E)$ induced by the action of $\mathrm{Gal}(\bar{K}/K)$ on the ℓ^n-torsion points of E. Then we have a natural representation σ_ℓ of $\mathrm{Gal}(\bar{K}/K)$ on $V_\ell(E) = T_\ell(E) \otimes \mathbb{Q}_\ell$. Let

$$\sigma'_\ell : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}(V_\ell(E)^*) \tag{2.91}$$

be the *contragredient* of σ_ℓ. Let $\sigma'_{\ell,\iota}$ be the associated complex two-dimensional representation of $W'(\bar{K}/K)$ as described in the previous section. One can show that, up to isomorphism, $\sigma'_{\ell,\iota}$ is independent of the choice of ℓ and ι. We will therefore write σ'_E for $\sigma'_{\ell,\iota}$ and call it the *Weil-Deligne representation* attached to E/K.

The case of good reduction.

Let E/K be an elliptic curve with good reduction. Let ℓ be a prime different from the residual characteristic of K, and let σ'_ℓ be the ℓ-adic representation of $\mathrm{Gal}(\bar{K}/K)$ on E. By Theorem 2.8.5, since E/K has good reduction, the action of $\mathrm{Gal}(\bar{K}/K)$ on $T_\ell(E)$ is unramified. Hence, the action of $W'(\bar{K}/K)$ on $T_\ell(E)$ is completely determined by $\sigma'_\ell(\Phi)$, where Φ is an inverse Frobenius element. Now we will calculate $\det(\sigma'_\ell(\Phi))$ and $\mathrm{tr}(\sigma'_\ell(\Phi))$ in order to find the characteristic polynomial of $\sigma'_\ell(\Phi)$.

The determinant of Frobenius: For a non-negative integer n, consider the group $W(\ell^n)$ of ℓ^n-th roots of unity in $\bar{K}$. Via the maps $W(\ell^{n+1}) \to W(\ell^n)$ defined as $\zeta \mapsto \zeta^\ell$, these groups form a projective system. The projective limit

$$T_\ell(K) := \varprojlim_n W(\ell^n), \tag{2.92}$$

is called the **Tate module** of K. It is a $\mathrm{Gal}(\bar{K}/K)$-module in a natural way. Since ℓ is different from the residual characteristic of K, we have $W(\ell^n) \cong \frac{\mathbb{Z}}{\ell^n\mathbb{Z}}$ and $T_\ell(K) \cong \mathbb{Z}_\ell$. Hence, the Galois action gives rise to a character $\chi_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_\ell^\times$. This is called the *cyclotomic character*. By definition,

$$g(\zeta) = \zeta^{\chi_\ell(g)} \quad \text{for all } \zeta \in T_\ell(K) \text{ and } g \in \mathrm{Gal}(\bar{K}/K). \tag{2.93}$$

Since the field extension $K(W(\ell^n))/K$ is unramified, the inertia group I_K acts trivially on $T_\ell(K)$, i.e., χ_ℓ is an unramified representation. Since $\Phi \in \mathrm{Gal}(\bar{K}/K)$ induces the inverse of the Frobenius map $x \to x^q$ on $\bar{k}$ and the Galois group of the extension $K(W(\ell^n))/K$ is generated by an element with the property $\zeta \to \zeta^q$, it is easy to see that

$$\chi_\ell(\Phi) = q^{-1}. \tag{2.94}$$

Now for the elliptic curve E/K, the *Weil pairing* is a bilinear, non-degenerate, Galois-invariant map

$$T_\ell(E) \times T_\ell(E) \longrightarrow T_\ell(K). \tag{2.95}$$

This pairing induces an isomorphism of $\mathrm{Gal}(\bar{K}/K)$ modules $\bigwedge^2 T_\ell(E) \cong T_\ell(K)$. The action of $\mathrm{Gal}(\bar{K}/K)$ on the left hand side is the determinant of the Galois representation σ'_ℓ on E. Hence we get $\det \circ \sigma_\ell = \chi_\ell$, and consequently $\det \circ \sigma'_\ell = \chi_\ell^{-1}$, where we recall that σ'_ℓ denotes the contragredient of σ_ℓ. Then from (2.94), we get

$$\det(\sigma'_\ell(\Phi)) = q. \tag{2.96}$$

The trace of Frobenius: Let $\tilde{E}/k$ be the reduced curve. By our assumption of good reduction, $\tilde{E}$ is non-singular. Corollary II.6.4 and Proposition VII.3.1 of [48] therefore imply that the reduction map $E \to \tilde{E}$ induces an isomorphism $E[\ell^n] \cong \tilde{E}[\ell^n]$ of the ℓ^n-division points. In fact, these isomorphism, for all n, fit together to produce an isomorphism of the Tate modules

$$T_\ell(E) \cong T_\ell(\tilde{E}) \quad \text{and} \quad V_\ell(E) \cong V_\ell(\tilde{E}). \tag{2.97}$$

This isomorphism is Galois-invariant in the sense that if $g \in \mathrm{Gal}(\bar{K}/K)$ induces the map $\tilde{g} \in \mathrm{Gal}(\bar{k}/k)$, then the reduction of $\sigma'_{E,\ell}(g)(v)$ equals $\sigma'_{\tilde{E},\ell}(\tilde{g})(\tilde{v})$ for all $v \in V_\ell(E)$. In particular,

$$\mathrm{tr}\left(\sigma'_{E,\ell}(g)\right) = \mathrm{tr}\left(\sigma'_{\tilde{E},\ell}(\tilde{g})\right) \text{ for all } g \in \mathrm{Gal}(\bar{k}/k), \tag{2.98}$$

here both sides are elements of $\mathbb{Q}_\ell$. Now, let $g = \Phi$. Then, by definition, $\tilde{\Phi}$ is the inverse of the Frobenius map $x \to x^q$ on $\bar{k}$. Let ϕ be the Frobenius isogeny on $\tilde{E}$ (i.e., the map $(x, y) \to (x^q, y^q))$). Then $\sigma'_{\tilde{E},\ell}(\tilde{\Phi}^{-1})$ is ϕ_ℓ in the notation of [48], Proposition V.2.3 and we get $\mathrm{tr}(\phi_\ell) = 1 + \deg(\phi) - \deg(1 - \phi)$. By Proposition II.2.11 of [48], $\deg(\phi) = q$. As in the proof of Theorem V.1.1 of [48], we have $\deg(1 - \phi) = \#\tilde{E}(k)$, the number of k-rational points on the reduced curve. Now, $\sigma'_{\tilde{E},\ell}(\tilde{\Phi}^{-1})$ and $\sigma'_{\tilde{E},\ell}(\tilde{\Phi})$ have the same trace since their matrix forms are conjugate of each other. Hence, using (2.98), we get

$$\mathrm{tr}\left(\sigma'_{E,\ell}(\Phi)\right) = \mathrm{tr}\left(\sigma'_{\tilde{E},\ell}(\tilde{\Phi})\right) = \mathrm{tr}\left(\sigma'_{\tilde{E},\ell}(\tilde{\Phi}^{-1})\right) = 1 + q - \#\tilde{E}(k). \tag{2.99}$$

Now, using (2.96) and (2.99), we get that the characteristic polynomial f_E of $\sigma'_{E,\ell}(\Phi)$ is $f_E = 1 - aX + qX^2$, where $a = 1 + q - \#\tilde{E}(k)$. Note that f_E has coefficients in $\mathbb{Z}$, which is not evident from its definition. Let $f_E = (1 - \alpha X)(1 - \beta X)$. Then α, β are algebraic integers such that $\alpha + \beta = a$ and $\alpha\beta = q$. So, $|\alpha| = |\beta| = \sqrt{q}$. It also follows that $\alpha = \bar{\beta}$.

Now, let $\sigma'_E = (\sigma_E, 0)$ be the Weil-Deligne representation attached to $\sigma'_{E,\ell}$. Then

σ'_E is unramified, since $\sigma'_{E,\ell}$ is, and σ'_E have the same characteristic polynomial as $\sigma'_{E,\ell}$. Summarizing, we obtain the following result.

Proposition 2.8.12. *Let E/K be an elliptic curve with good reduction. Let $\tilde{E}/k$ be the reduced curve. Then the associated Weil-Deligne representation σ'_E is of the form $(\sigma_E, 0)$, where (up to isomorphism) the representation σ_E of $W(\bar{K}/K)$ is given as follows:*

1. *σ_E is unramified, i.e., $\sigma_E(I_K) = 0$.*
2. *$\sigma_E(\Phi) = \begin{bmatrix} \beta & \\ & \bar{\beta} \end{bmatrix}$, where β is an algebraic integer with $|\beta| = \sqrt{q}$ and $\beta + \bar{\beta} = a$. Here, $a = 1 + q - \#\tilde{E}(k)$.*

L-functions of elliptic curves.

The L-function (in arithmetic normalization) of an elliptic curve $E/\mathbb{Q}$ is given by

$$L^{\mathrm{ar}}(s, E) = \prod_p L_p^{\mathrm{ar}}(s, E), \tag{2.100}$$

where $L_p^{\mathrm{ar}}(s, E)$ is the L-function (in arithmetic normalization) of the elliptic curve $E/\mathbb{Q}_p$ defined as

$$L_p^{\mathrm{ar}}(s, E) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}, \quad \text{where } a_p = 1 + p - \#\tilde{E}(\mathbb{F}_p). \tag{2.101}$$

We consider the L-function $L(s, E)$ of $E/\mathbb{Q}$ in analytic normalization given by $L(s, E) := L^{\mathrm{ar}}(s - 1/2, E) = \prod_p L_p(s, E)$. If $\sigma'_{E/\mathbb{Q}_p}$ is the Weil-Deligne representation attached to $E/\mathbb{Q}_p$, then we have $L_p(s, E) = L_p(s, \sigma'_{E/\mathbb{Q}_p})$, where $L_p(s, \sigma'_{E/\mathbb{Q}_p})$ is the L-function of $\sigma'_{E/\mathbb{Q}_p}$ as in (2.42). Let $\pi = \otimes_{p \leq \infty} \pi_p$ be the automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ associated to $E/\mathbb{Q}$. Then the local representation π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ has the L-parameter $\sigma'_{E/\mathbb{Q}_p}$. So, we have the following identity

$$L(s, E) = L(s, \pi), \text{ i.e.,} \quad L_p(s, E) = L_p(s, \sigma'_{E/\mathbb{Q}_p}) = L_p(s, \pi_p). \tag{2.102}$$

Also, by the modularity theorem, there exists a modular form ϕ of weight 2 associated with $E/\mathbb{Q}$ such that $L(s,E) = L(s,\phi)$, where $L(s,\phi)$ is defined as in (2.62).

2.8.5 Kodaira-Néron types

Again, let us assume that K is a non-archimedean local field of characteristic zero with residual characteristic p and k is the residue field of K of order q. In this section we discuss some facts about the Kodaira-Néron types of elliptic curves over K. One of the main result that we study in this section is that the Kodaira-Néron types of an elliptic curve E/K can be determined from its Weierstrass equation. We also study the relationship between the Kodaira-Néron type and the reduction type of an elliptic curve E/K. Our primary references are Section C.15 of [48] and Chapter IV of [47].

Néron model and Tate's algorithm.

Let E/K be an elliptic curve given by a minimal Weierstrass equation of the form (2.71) with coefficients in $\mathfrak{o}_K$. This equation can be used to define a scheme over $\mathrm{Spec}(\mathfrak{o}_K)$. The resulting scheme may not be nonsingular if E has bad reduction. By resolving the singularity, we obtain a (smooth) group scheme $\mathcal{E}/\mathfrak{o}_K$ whose generic fiber is E/K and it satisfies a universal property known as the *Néron mapping property*. We call $\mathcal{E}/\mathfrak{o}_K$ a **Néron model** of E/K. Note that E/K always has a *minimal proper regular model* $\mathcal{C}/\mathfrak{o}_K$ defined as in Theorem 4.5 of [47]. Then, by Theorem 6.1 of [47], the Néron model $\mathcal{E}/\mathfrak{o}_K$ of an elliptic curve E/K exists and it is the largest subscheme of $\mathcal{C}/\mathfrak{o}_K$ which is smooth over $\mathfrak{o}_K$. The generic fiber of $\mathcal{C}$ is E and the special fiber $\tilde{\mathcal{C}} = \mathcal{C} \times_{\mathrm{Spec}(\mathfrak{o}_K)} \mathrm{Spec}(k)$ of $\mathcal{C}$ consists of one or more irreducible components with multiplicity. Then $\mathcal{E}$ is constructed by removing from $\mathcal{C}$ all the irreducible components with multiplicity ≥ 2, all the singular points on each component, and all the points where these components intersect (see Remark 6.1.1 in [47] and Theorem 15.1 in [48]).

Consider the reduced curve $\tilde{E}/k$ and the set of nonsingular points $\tilde{E}_{\mathrm{ns}}(k)$ as in Section 2.8.2. We define $E_0(K) = \left\{P \in E(K) : \tilde{P} \in \tilde{E}_{\text{ns}}(k)\right\}$. The classification of special fibers $\tilde{\mathcal{C}}$ helps to determine the special fiber $\tilde{\mathcal{E}} = \mathcal{E} \times_{\text{Spec}(\mathfrak{o}_K)} \text{Spec}(k)$ of $\mathcal{E}$ and hence the group $E(K)/E_0(K)$ (see Theorem 15.1 in [48]). Kodaira and Néron have classified all possibilities for the special fiber $\tilde{\mathcal{C}}$ and $E(K)/E_0(K)$ associated to a given elliptic curve E/K and it is known as the **Kodaira-Néron type** of E/K. See Theorem 8.2 and Table 4.1 of [47] for all the possible Kodaira-Néron types.

Tate has an algorithm, known as **Tate's algorithm**, to compute the special fiber $\tilde{\mathcal{C}}$, i.e., the Kodaira-Néron type of a given elliptic curve E/K in terms of its Weierstrass equation. See 9.4 from Section 9 of [47] for the detailed algorithm described in eleven steps. Details of the algorithm are not needed for our study. But, we use the Kodaira-Néron type of an elliptic curve E/K given in terms of its Weierstrass equation, in order to compute the Weil-Deligne representation σ'_E attached to E/K in Chapter 4.

Relationship between the Kodaira-Néron type and the reduction type.

We have discussed the reduction types of elliptic curves E over K in Section 2.8.2. Now, we will see how the reduction type and the Kodaira-Néron type of E/K are related. We relate these "two different Types" attached to E/K using their description given in terms of the Weierstrass coefficients of E/K. Here we state the results without proofs, we refer to Sections 9-11 of [47], and [33] for details.

1. E/K has **good reduction** if and only if the Kodaira-Néron type of E/K is **Type** I_0.

2. E/K has **multiplicative reduction** if and only if the Kodaira-Néron type of E/K is **Type** I_n with $n \geq 1$.

3. E/K has **potential multiplicative reduction** with the residual characteristic of K being odd if and only if the Kodaira-Néron type of E/K is **Type** I_n^* with $n \geq 1$.

4. E/K has **potential good reduction** with the residual characteristic of K being odd if and only if the Kodaira-Néron type of E/K is **one of the Types** I_0^*, II, II^*, III, III^*, IV or IV^*.

Chapter 3

The symmetric cube lifting

In this chapter we introduce the symmetric cube map and discuss the lifting of automorphic representations of GL(2) via the symmetric cube map. We call this "the symmetric cube lifting". We also discuss the local symmetric cube lifting in Section 3.3 and Siegel modular forms coming from this lifting in Section 3.5.

3.1 The symmetric cube map

Let V be the space of homogeneous polynomials of degree 3 in $\mathbb{C}[S,T]$. Then V has a basis $\{S^3, S^2T, ST^2, T^3\}$. There is an action of $\mathrm{GL}(2,\mathbb{C})$ on V given by

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot f(S,T) = f(aS+cT, bS+dT).$$

This action defines a four-dimensional irreducible representation of $\mathrm{GL}(2,\mathbb{C})$ and induces a map $f : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(V) \cong \mathrm{GL}(4,\mathbb{C})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{f}{\longmapsto} \begin{bmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & 2abc+a^2d & 2abd+b^2c & 3b^2d \\ 3ac^2 & 2acd+bc^2 & 2bcd+ad^2 & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{bmatrix}.$$

The image of f does not lie in $\mathrm{GSp}(4,\mathbb{C})$ since

$$f(\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]) = \begin{bmatrix} 1 & b & b^2 & b^3 \\ & 1 & 2b & 3b^2 \\ & & 1 & 3b \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & b & & \\ & 1 & & \\ & & 1 & 3b \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -b^2 & -2b^3 \\ & 1 & 2b & 3b^2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \notin \mathrm{GSp}(4,\mathbb{C}),$$

and $f\left(\left[\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right]\right) = \begin{bmatrix} 1 & & & \\ 3c & 1 & & \\ 3c^2 & 2c & 1 & \\ c^3 & c^2 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 3c & 1 & & \\ & & 1 & \\ & & c & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ 3c^2 & 2c & 1 & \\ -2c^3 & -c^2 & & 1 \end{bmatrix} \notin \mathrm{GSp}(4,\mathbb{C})$.

Note 3.1.1. *Conjugating $f\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right)$ by a element of $\mathrm{GL}(4,\mathbb{C})$ gives us the same representation up to isomorphism and we can find an element A of $\mathrm{GL}(4,\mathbb{C})$ such that the image of $A \cdot f\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) \cdot A^{-1}$ lies in $\mathrm{GSp}(4,\mathbb{C})$.*

Let us choose $A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{bmatrix} \in \mathrm{GL}(4,\mathbb{C})$. Then we get the following

$$
\begin{aligned}
(A \cdot f \cdot A^{-1})\left(\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]\right) &= \begin{bmatrix} 1 & b & & \\ & 1 & & \\ & & 1 & -b \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -b^2 & -\frac{2}{3}b^3 \\ & 1 & 2b & -b^2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C}), \\
(A \cdot f \cdot A^{-1})\left(\left[\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right]\right) &= \begin{bmatrix} 1 & & & \\ 3c & 1 & & \\ & & 1 & \\ & & -3c & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ 3c^2 & 2c & 1 & \\ 6c^3 & 3c^2 & & 1 \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C}).
\end{aligned}
$$

Using the identity $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & \\ ca^{-1} & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a^{-1}b \\ & 1 \end{smallmatrix}\right]$, we get the following version of the symmetric cube map, denoted as sym^3, and given by

$$
\begin{aligned}
&\mathrm{sym}^3 : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GSp}(4,\mathbb{C}) \\
&\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^3 & a^2b & ab^2 & -\frac{1}{3}b^3 \\ 3a^2c & 2abc + a^2d & 2abd + b^2c & -b^2d \\ 3ac^2 & 2acd + bc^2 & 2bcd + ad^2 & -bd^2 \\ -3c^3 & -3c^2d & -3cd^2 & d^3 \end{bmatrix}.
\end{aligned} \tag{3.1}
$$

Note that, if $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{C})$ then $\mathrm{sym}^3(g) \in \mathrm{Sp}(4,\mathbb{C})$. Also, since we use (3.1) for calculations over non-achimedean local fields of characteristic zero, here we choose the symplectic form $J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$ for simplicity of calculations.

3.2 Langlands functoriality for the symmetric cube map

Here, a "lifting" means a functorial lift according to *the Langlands principle of functoriality* discussed in Section 2.6.2. We have the map $\mathrm{sym}^3 : \widehat{\mathrm{GL}(2)} = \mathrm{GL}(2,\mathbb{C}) \to \widehat{\mathrm{GSp}(4)} = \mathrm{GSp}(4,\mathbb{C})$, which is a Lie group homomorphism. So the principle of functoriality predicts that an automorphic representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ should "lift" to an automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$, i.e., the following diagram should hold:

$$\left\{\ \mathrm{GL}(2,\mathbb{C}) \xrightarrow{\mathrm{sym}^3} \mathrm{GSp}(4,\mathbb{C})\ \right\} \xRightarrow[\text{Functoriality}]{\text{Langlands}} \left\{ \begin{array}{ccc} \text{auto. rep. of} & & \text{auto. rep. of} \\ \mathrm{GL}(2,\mathbb{A}_\mathbb{Q}) & \xrightarrow[\text{lifting}]{\mathrm{sym}^3} & \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q}) \end{array} \right\}. \tag{3.2}$$

Ramakrishnan and Shahidi [35] proved the following lifting from a cuspidal automorphic representation $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ to a cuspidal automorphic representation $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$.

Theorem 3.2.1 (Ramakrishnan-Shahidi, 2007)**.** *Let $\pi \cong \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ defined by a holomorphic, non-CM newform ϕ of even weight $k \geq 2$ and level N with trivial central character. Then there exists a cuspidal automorphic representation $\Pi \cong \bigotimes_p \Pi_p$ of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ with trivial central character, which is unramified at any prime p not dividing N, such that*

(i) *Each non-archimedean component Π_p is generic, with its parameter being sym^3 of the parameter of π_p.*

(ii) *Π_∞ is a holomorphic discrete series representation, with its parameter being sym^3 of the archimedean parameter of π.*

(iii) $L(s,\Pi) = L(s,\pi,\mathrm{sym}^3)$.

Using some results that were not available at the time, we can now give a short proof of Theorem 3.2.1 as follows:

Proof. By the functorial symmetric cube transfer for cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ from [19], we get a unitary cuspidal automorphic representation $\mu = \otimes_p \mu_p$ of $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character from π such that $L(s, \mu) = L(s, \pi, \mathrm{sym}^3)$. Also, μ has the following properties:

- μ is symplectic, i.e., the exterior square L-function $L(s, \mu, \Lambda^2)$ has a pole at $s = 1$. This is true because of the well known identity $L(s, \mu, \Lambda^2) = L(s, \pi, \mathrm{sym}^4)\zeta(s)$ and the fact that $L(s, \pi, \mathrm{sym}^4)$ has no zero at $s = 1$.

- μ is self-dual. This follows from the identity $L(s, \mu \times \mu) = L(s, \mu, \Lambda^2)L(s, \mu, \mathrm{sym}^2)$ and the facts that $L(s, \mu, \mathrm{sym}^2)$ has no zero at $s = 1$ and the Rankin-Selberg L-function has a pole at $s = 1$ if and only if μ is isomorphic to its contragredient $\mu^\vee$.

These two properties are explained in detail in [35]. Now, since μ is a self-dual, symplectic, unitary, cuspidal automorphic representation of $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$, $\psi = \mu \boxtimes 1$ is an Arthur parameter of general type for the group $\mathrm{SO}(5)$. Then by Arthur's classification in [2], there is a packet Π_ψ of cuspidal automorphic representations of $\mathrm{SO}(5, \mathbb{A}_{\mathbb{Q}})$ with ψ as the Arthur parameter. Since $\mathrm{SO}(5)$ and $\mathrm{PGSp}(4)$ are isomorphic as algebraic groups, a representation of $\mathrm{SO}(5, \mathbb{A}_{\mathbb{Q}})$ can be viewed as a representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character. Furthermore, using Proposition 1.2.1 of [44], it follows that there exists an element $\Pi \cong \otimes \Pi_p$ in the packet Π_ψ such that each non-archimedean local representation Π_p is a generic representation and Π_∞ is a holomorphic discrete series representation. We choose such a representation Π in Π_ψ. So, we get a cuspidal automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character which satisfies properties (i) and (ii) of Theorem 3.2.1. Also, it is evident from Lemma 1.3.1 of [44] that the spin L-function of Π coincides with the standard L-function $L(s, \mu)$, i.e., $L(s, \Pi) = L(s, \mu) = L(s, \pi, \mathrm{sym}^3)$. $\square$

Moreover, this sym^3 lifting (3.2) is functorial at each place since the local Langlands correspondence is true for $\mathrm{GL}(2, \mathbb{Q}_p)$ and $\mathrm{GSp}(4, \mathbb{Q}_p)$. So, we have the following diagram

$$\pi_p \xrightarrow{\text{LLC}} \left[\varphi_p \colon W'_{\mathbb{Q}_p} \to \mathrm{GL}(2, \mathbb{C})\right] \xrightarrow{\mathrm{sym}^3} \left[\mathrm{sym}^3 \circ \varphi_p \colon W'_{\mathbb{Q}_p} \to \mathrm{GSp}(4, \mathbb{C})\right] \xrightarrow{\text{LLC}} \Pi_p.$$

We can specifically study the "local components" of the sym^3 lifting, i.e., for an irreducible and admissible representation π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$, we can consider the representations Π_p of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}_p})$ whose L-parameter is the same as the sym^3 of the L-parameter of π_p. For the purposes of this paper, it is important to study the "local sym^3 lift" at each place, which we discuss in the next two sections.

3.3 Local sym^3 lifting of non-archimedean parameter of $\mathrm{GL}(2)$

Let K be a non-achimedean local field of characteristic 0 and residual characteristic p, and let v be the valuation on K. We denote by $\mathfrak{o}_K$ the ring of integers of K, and we let p be the maximal ideal of $\mathfrak{o}_K$. Let q be the number of elements of $\mathfrak{o}_K/\mathfrak{p}$. We fix a generator ϖ_K for the ideal $\mathfrak{p}$ and ϖ_K is called the uniformizer. We write $\nu(x)$ or $|x|$ for the normalized absolute value of x; thus $\nu(\varpi_K) = q^{-1}$. Let π be an infinite dimensional irreducible admissible representation of $\mathrm{GL}(2, K)$. In this chapter, we look at the sym^3 lifting of π. We consider $\mathrm{sym}^3(\pi)$ as the L-packet on $\mathrm{GSp}(4, K)$ whose L-parameter is the symmetric cube of the L-parameter of π.

Langlands parameter of $\mathrm{sym}^3(\pi)$

There are three types of irreducible admissible infinite dimensional representations of $\mathrm{GL}(2, K)$. So, we compute the L-parameter of $\mathrm{sym}^3(\pi)$ in three cases.

Case 1: Let $\pi = \chi_1 \times \chi_2$ be the principal series representation of $\mathrm{GL}(2, K)$, where χ_1, χ_2 are characters of $K^\times$ such that $\chi_1\chi_2^{-1} \neq \nu^{\pm 1}$. Then, using the L-parameter of π in (2.40) and the sym^3 map in (3.1), we get the L-parameter $(\mathrm{sym}^3(\varphi), N')$ of $\mathrm{sym}^3(\pi)$ as follows

$$\mathrm{sym}^3(\varphi)(w) = \begin{bmatrix} \chi_1^3(w) & & & \\ & \chi_1^2\chi_2(w) & & \\ & & \chi_1\chi_2^2(w) & \\ & & & \chi_2^3(w) \end{bmatrix}, w \in W(\bar{K}/K) \text{ and } N' = 0 \tag{3.3}$$

Note 3.3.1. *Let us assume that* $\pi = \chi_1 \times \chi_2$ *is a local component of a cuspidal automorphic representation on* GL(2)*. Then, by the work of Kim and Shahidi in [18], a well-known bound on the exponent of the character* $\chi_1\chi_2^{-1}$ *is* $\frac{2}{9}$*. Hence,* $(\chi_1\chi_2^{-1})^2 \neq \nu^{\pm 1}$ *and* $(\chi_1\chi_2^{-1})^3 \neq \nu^{\pm 1}$.

When $\pi = \chi_1 \times \chi_2$ is associated to a cuspidal automorphic representation on GL(2), using Note 3.3.1 and the local Langlands correspondence for $\mathrm{GSp}(4, K)$, we get

$$\mathrm{sym}^3(\pi) = \chi_1^2\chi_2^{-2} \times \chi_1\chi_2^{-1} \rtimes \chi_2^3,$$

which is a **type I** representation of $\mathrm{GSp}(4, K)$ (see Section 2.4 of [37]).

Case 2: Let $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)}$ be the (twisted) Steinberg representation of $\mathrm{GL}(2, K)$, where χ is a character of $K^\times$. The L-parameter (φ, N) of π is as in (2.41). Using the sym^3 map (3.1), we can find the L-parameter $(\mathrm{sym}^3(\varphi), N')$ of $\mathrm{sym}^3(\pi)$. Note that,

$$\mathrm{sym}^3(\varphi)(w) = \begin{bmatrix} \chi^3(w)|w|^{\frac{3}{2}} & & & \\ & \chi^3(w)|w|^{\frac{1}{2}} & & \\ & & \chi^3(w)|w|^{-\frac{1}{2}} & \\ & & & \chi^3|w|^{-\frac{3}{2}} \end{bmatrix}, w \in W(\bar{K}/K)$$

To calculate the nilpotent part N' of the L-parameter of $\mathrm{sym}^3(\pi)$, we note that $N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{gl}(2, \mathbb{C})$, where $\mathfrak{gl}(2, \mathbb{C})$ is the Lie algebra of $\mathrm{GL}(2, \mathbb{C})$. Here we have the following diagram

$$\begin{array}{ccc} \exp(tN) \in \mathrm{GL}(2, \mathbb{C}) & \xrightarrow{\mathrm{sym}^3} & \mathrm{GL}(4, \mathbb{C}) \\ {\scriptstyle \exp}\big\uparrow & & \big\downarrow{\scriptstyle d(\mathrm{sym}^3)|_{t=0}} \\ N \in \mathfrak{gl}(2, \mathbb{C}) & \longrightarrow & \mathfrak{gsp}(4, \mathbb{C}) \end{array}$$

Now, $\exp(tN) = \left[\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{C})$ and $\mathrm{sym}^3(\exp(tN)) = \begin{bmatrix} 1 & t & t^2 & -\frac{1}{3}t^3 \\ & 1 & 2t & -t^2 \\ & & 1 & -t \\ & & & 1 \end{bmatrix}$. So, we get

$$N' = d\left(\mathrm{sym}^3(\exp(tN))\right)|_{t=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gsp}(4,\mathbb{C}).$$

Using the identity $\begin{bmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we get the L-parameter of $\mathrm{sym}^3(\pi)$ as follows

$$\mathrm{sym}^3(\varphi)(w) = \begin{bmatrix} \chi^3(w)|w|^{\frac{3}{2}} & & & \\ & \chi^3(w)|w|^{\frac{1}{2}} & & \\ & & \chi^3(w)|w|^{-\frac{1}{2}} & \\ & & & \chi^3|w|^{-\frac{3}{2}} \end{bmatrix}, w \in W(\bar{K}/K),\ \ N' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

Then, using the local Langlands correspondence for $\mathrm{GSp}(4,K)$, we get

$$\mathrm{sym}^3(\pi) = \chi^3\mathrm{St}_{\mathrm{GSp}(4,K)},$$

which is of **type IVa** (see Section 2.4 of [37]).

Case 3: Let π be a supercuspidal representation of $\mathrm{GL}(2,K)$. When the residual characteristic of K is odd, we have $\pi = \omega_{F,\xi}$, a dihedral supercuspidal representation, where F/K is a quadratic extension and ξ is a character of $F^\times$ and $\xi \neq \xi^\sigma$, σ is the nontrivial element in $\mathrm{Gal}(F/K)$. Using the L-parameter of π in (2.35) and the sym^3 map (3.1), we get the L-parameter of $\mathrm{sym}^3(\pi)$ as follows

$$\mathrm{sym}^3(\varphi)(w) = \begin{bmatrix} \xi^3(w) & & & \\ & \xi^2\xi^\sigma(w) & & \\ & & \xi(\xi^\sigma)^2(w) & \\ & & & (\xi^\sigma)^3(w) \end{bmatrix},$$

$$\mathrm{sym}^3(\varphi)(\sigma) = \begin{bmatrix} & & & -\frac{1}{3} \\ & & (\xi^\sigma)(\sigma^2) & \\ & (\xi^\sigma)^2(\sigma^2) & & \\ -3(\xi^\sigma)^3(\sigma^2) & & & \end{bmatrix}.$$

This parameter is a sum of the 2-dimensional representations given as follows

$$w \mapsto \begin{bmatrix} \xi^3(w) & \\ & (\xi^\sigma)^3(w) \end{bmatrix}, \qquad \sigma \mapsto \begin{bmatrix} & -\frac{1}{3} \\ -3(\xi^\sigma)^3(\sigma^2) & \end{bmatrix},$$

$$\text{and} \quad w \mapsto \begin{bmatrix} \xi^2\xi^\sigma(w) & \\ & \xi(\xi^\sigma)^2(w) \end{bmatrix}, \qquad \sigma \mapsto \begin{bmatrix} & (\xi^\sigma)(\sigma^2) \\ (\xi^\sigma)^2(\sigma^2) & \end{bmatrix}. \tag{3.5}$$

Since

$$\begin{bmatrix} 1 & \\ & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} & -\frac{1}{3} \\ -3(\xi^\sigma)^3(\sigma^2) & \end{bmatrix} \begin{bmatrix} 1 & \\ & -3 \end{bmatrix} = \begin{bmatrix} & 1 \\ (\xi^\sigma)^3(\sigma^2) & \end{bmatrix},$$

$$\begin{bmatrix} 1 & \\ & (\xi^\sigma)(\sigma^2) \end{bmatrix} \begin{bmatrix} & (\xi^\sigma)(\sigma^2) \\ (\xi^\sigma)^2(\sigma^2) & \end{bmatrix} \begin{bmatrix} 1 & \\ & (\xi^\sigma)^{-1}(\sigma^2) \end{bmatrix} = \begin{bmatrix} & 1 \\ (\xi^\sigma)^3(\sigma^2) & \end{bmatrix},$$

and $(\xi^2\xi^\sigma)(\sigma^2) = \xi^2(\sigma^2)\xi^\sigma(\sigma^2) = (\xi^\sigma)^2(\sigma^2)\xi^\sigma(\sigma^2) = (\xi^\sigma)^3(\sigma^2)$, we get that the L-parameter of $\mathrm{sym}^3(\pi)$ as follows

$$\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3) \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma). \tag{3.6}$$

We denote its L-packet $\mathrm{sym}^3(\omega_{F,\xi})$ as $\omega_{F,\xi^3} \oplus \omega_{F,\xi^2\xi^\sigma}$.

Now, we determine the representation type of $\mathrm{sym}^3(\omega_{F,\xi})$. We use the following remark.

Remark 3.3.2. $\mathrm{ind}_{W_F}^{W_K}(\xi_1) \cong \mathrm{ind}_{W_F}^{W_K}(\xi_2)$ *if and only if* $\xi_1 = \xi_2$ *or* $\xi_1 = \xi_2^\sigma$.

Firstly, note that, $\xi^2\xi^\sigma = (\xi^2\xi^\sigma)^\sigma = (\xi^\sigma)^2\xi$ if and only if $\xi = \xi^\sigma$. Since $\xi \neq \xi^\sigma$, we get that $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$ is always irreducible by Remark 3.3.2. So, we only need to check whether $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is reducible or irreducible. We have the following three cases:

(1) $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is irreducible and isomorphic to $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$.

(2) $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is reducible.

(3) $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is irreducible and not isomorphic to $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$.

Case (1): Assume that $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is irreducible and isomorphic to $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$. By Remark 3.3.2, this happens if and only if either $\xi^3 = \xi^2\xi^\sigma$ or $\xi^3 = (\xi^2\xi^\sigma)^\sigma$ on $F^\times$.

Now, $\xi^3 = \xi^2\xi^\sigma$ if and only if $\xi = \xi^\sigma$, which is not true by our assumption. So, $\xi^3 = (\xi^2\xi^\sigma)^\sigma$, i.e., $\xi^2 = (\xi^2)^\sigma$ on $F^\times$ if and only if $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3) \cong \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$. By the local Langlands correspondence for $\mathrm{GSp}(4,K)$, in this case the L-packet $\mathrm{sym}^3(\omega_{F,\xi})$ is of **type VIII**. The representations in the L-packet $\mathrm{sym}^3(\omega_{F,\xi})$ are $\tau(S,\omega_{F,\xi})$ (type VIIIa) and $\tau(T,\omega_{F,\xi})$ (type VIIIb) (see Sections 2.2 and 2.4 of [37]).

Moreover when $a(\pi) = 2$, the condition $\xi^2 = (\xi^\sigma)^2$ implies the representation $\omega_{F,\xi}$ is triply imprimitive (see Corollary 3.2 from [45]). So, $\mathrm{sym}^3(\omega_{F,\xi})$ is of type VIII if $\omega_{F,\xi}$ is a **triply imprimitive** representation with $a(\omega_{F,\xi}) = 2$.

Case (2): Assume that $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ is reducible. By Remark 3.3.2, this happens if and only if $\xi^3 = (\xi^3)^\sigma$. In this case, $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3) = \varphi \oplus \varphi\chi_{F/K}$, for some character $\varphi : K^\times \to \mathbb{C}^\times$ and the unique quadratic character $\chi_{F/K}$ attached to F/K. Note that, ξ^3 factors through the norm $N_{F/K}$ and $\xi^3 = \varphi \circ N_{F/K}$. So, the local parameter of $\mathrm{sym}^3(\omega_{F,\xi})$ is of the form $\varphi\chi_{F/K} \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma) \oplus \varphi$. Now, the central character of $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$ equals $\xi^2\xi^\sigma|_{K^\times} \cdot \chi_{F/K} = \xi^3|_{K^\times} \cdot \chi_{F/K} = \varphi^2 \cdot \chi_{F/K}$ (since $\xi^3 = \varphi \circ N_{F/K} \Leftrightarrow \xi^3|_{K^\times} = \varphi^2$, because $\xi^3(x) = \varphi \circ N_{F/K}(x) = \varphi(x^2) = \varphi^2(x)$ for $x \in K^\times$). So, the L-parameter of $\mathrm{sym}^3(\omega_{F,\xi})$ is of the form

$$w \mapsto \begin{bmatrix} \varphi\chi_{F/K}(w) & & \\ & \varphi\mu(w) & \\ & & \varphi(w) \end{bmatrix}, \quad w \in W(\bar{K}/K)$$

such that $\varphi\mu = \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$, where μ is an irreducible L-parameter of $\mathrm{GL}(2,K)$ with $\det(\mu) = \chi_{F/K}$. Then μ is the L-parameter of the representation $\varphi^{-1} \otimes \omega_{F,\xi^2\xi^\sigma} = \omega_{F,\xi^2\xi^\sigma(\varphi^{-1}\circ N_{F/K})} = \omega_{F,\xi^2\xi^\sigma\xi^{-3}} = \omega_{F,\xi^{-1}\xi^\sigma}$. By the local Langlands correspondence for $\mathrm{GSp}(4,K)$, $\mathrm{sym}^3(\omega_{F,\xi}) = \omega_{F,\xi^{-1}\xi^\sigma} \rtimes \varphi$ is of **type X** (see Sections 2.2 and 2.4 of [37]).

Case (3): Let $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ be irreducible and not isomorphic to $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$. By Remark 3.3.2 and case (1), this happens if and only if $\xi^2 \neq (\xi^2)^\sigma$ and $\xi^3 \neq (\xi^3)^\sigma$ on

$F^\times$. In this case by local Langlands correspondence for $\mathrm{GSp}(4,K)$, $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3) \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^2\xi^\sigma)$ corresponds to a **supercuspidal** representation of $\mathrm{GSp}(4,K)$, i.e., $\mathrm{sym}^3(\omega_{F,\xi})$ is supercuspidal.

The $\mathrm{GL}(2,K)$ representations we consider for the sym^3 lifting have trivial central character. In Table 3.1, we list the L-parameter and the representation type of the L-packet $\mathrm{sym}^3(\pi)$, where π is representation of $\mathrm{GL}(2,K)$ with trivial central character. We also assume that π is a local component of a cuspidal automorphic representation on $\mathrm{GL}(2)$. When $\pi = \omega_{F,\xi}$ has trivial central character, by Remark 2.2.9, we have $\xi^\sigma = \xi^{-1}$.

Let π be a representation of $\mathrm{GL}(2,K)$ with trivial central character. When $p \geq 3$, using the local parameters in (3.3), (3.4), (3.6), and Table A.9. from [37], we describe the conductor $a(\mathrm{sym}^3(\pi))$ of $\mathrm{sym}^3(\pi)$ in Table 3.2.

3.4 Local sym^3 lifting of archimedean parameter of $\mathrm{GL}(2)$

To get the standard normalization at the archimedean place we should consider the classical version of $\mathrm{GSp}(4,\mathbb{C})$. So, in this case we consider the following version of the sym^3 map

$$\mathrm{sym}^3 : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GSp}(4,\mathbb{C})$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 2abc+a^2d & 3a^2c & 2abd+b^2c & -b^2d \\ a^2b & a^3 & ab^2 & -\frac{1}{3}b^3 \\ 2acd+bc^2 & 3ac^2 & 2bcd+ad^2 & -bd^2 \\ -3c^2d & -3c^3 & -3cd^2 & d^3 \end{bmatrix}. \tag{3.7}$$

Let π_∞ be the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ of lowest weight $k \geq 2$, i.e., $\pi_\infty = \mathcal{D}_k$. Then the L-parameter φ_k of π_∞ is given by

$$\varphi_k : W_{\mathbb{R}} \to \mathrm{GL}(2,\mathbb{C})$$

Table 3.1: L-parameter of $\mathrm{sym}^3(\pi)$ for local representations π of $\mathrm{GL}(2,K)$.

Here we denote $\mathrm{ind}_{W(\overline{K}/F)}^{W(\overline{K}/K)}(\xi)$ as $\mathrm{ind}_F^K \xi$. Also, μ is an irreducible L-parameter of $\mathrm{GL}(2,K)$ with $\det(\mu) = \chi_{F/K}$, and φ is a character of $K^\times$ such that $\xi^3 = \varphi \circ N_{F/K}$.

$\mathrm{GL}(2,K)$ representation π	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi)$	L-parameter of $\mathrm{sym}^3(\pi)$	Representation type of the L-packet $\mathrm{sym}^3(\pi)$
$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$\chi^3 \oplus \chi \oplus \chi^{-1} \oplus \chi^{-3}$	I
$\chi\mathrm{St}_{\mathrm{GL}(2)}$	$\chi^3\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{\frac{3}{2}}\chi^3 \oplus \nu^{\frac{1}{2}}\chi^3 \oplus \nu^{-\frac{1}{2}}\chi^3 \oplus \nu^{-\frac{3}{2}}\chi^3$	IVa
$\omega_{F,\xi}$ $\xi^4 = 1$ on $F^\times$	$\tau(S, \omega_{F,\xi})$, $\tau(T, \omega_{F,\xi})$	$\varphi_\pi \oplus \varphi_\pi$ $\varphi_\pi = \mathrm{ind}_F^K \xi$	VIII
$\omega_{F,\xi}$ $\xi^6 = 1$ on $F^\times$	$\omega_{F,\xi^4} \rtimes \varphi$	$\varphi\chi_{F/K} \oplus \varphi\mu \oplus \varphi$ $\varphi\mu = \mathrm{ind}_F^K \xi$	X
$\omega_{F,\xi}$ $\xi^4 \neq 1$ on $F^\times$ $\xi^6 \neq 1$ on $F^\times$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$	$\mathrm{ind}_F^K \xi \oplus \mathrm{ind}_F^K \xi^3$	Supercuspidal

Table 3.2: Conductor of $\mathrm{sym}^3(\pi)$ for local representations π of $\mathrm{GL}(2,K)$.

π	$\mathrm{sym}^3(\pi)$	Condition on π	$a(\pi)$	$a(\mathrm{sym}^3(\pi))$
$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$		$2a(\chi)$	$2a(\chi^3) + 2a(\chi)$
$\chi\mathrm{St}_{\mathrm{GL}(2)}$	$\chi^3\mathrm{St}_{\mathrm{GSp}(4)}$	χ is ram.	$2a(\chi)$	$4a(\chi^3)$
		χ is unr.	1	3
$\omega_{F,\xi}$	$\omega_{F,\xi^3} \oplus \omega_{F,\xi}$	F/K is unr.	$2a(\xi)$	$2a(\xi^3) + 2a(\xi)$
		F/K is ram.	$a(\xi) + 1$	$a(\xi^3) + a(\xi) + 2$

$$re^{i\theta} \mapsto \begin{bmatrix} e^{i(k-1)\theta} & \\ & e^{-i(k-1)\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} & (-1)^{k-1} \\ 1 & \end{bmatrix}. \tag{3.8}$$

Then, using the map in (3.7), we get

$$\begin{aligned} \mathrm{sym}^3(\varphi_k) : W_{\mathbb{R}} &\to \mathrm{GSp}(4, \mathbb{C}) \\ re^{i\theta} &\mapsto \begin{bmatrix} e^{i(k-1)\theta} &&& \\ & e^{i3(k-1)\theta} && \\ && e^{-i(k-1)\theta} & \\ &&& e^{-i3(k-1)\theta} \end{bmatrix} \\ j &\mapsto \begin{bmatrix} && (-1)^{2(k-1)} & \\ &&& -\frac{1}{3}(-1)^{3(k-1)} \\ (-1)^{(k-1)} &&& \\ & -3 && \end{bmatrix}. \end{aligned}$$

Now, we have the following identity

$$\begin{aligned} &\begin{bmatrix} (-1)^{\frac{(k-1)}{2}} &&& \\ & -\sqrt{3} && \\ && (-1)^{-\frac{(k-1)}{2}} & \\ &&& \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} && (-1)^{2(k-1)} & \\ &&& -\frac{1}{3}(-1)^{3(k-1)} \\ (-1)^{(k-1)} &&& \\ & -3 && \end{bmatrix} \begin{bmatrix} (-1)^{-\frac{(k-1)}{2}} &&& \\ & -\frac{1}{\sqrt{3}} && \\ && (-1)^{\frac{(k-1)}{2}} & \\ &&& -\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} && (-1)^{3(k-1)} & \\ &&& (-1)^{3(k-1)} \\ 1 &&& \\ & 1 && \end{bmatrix}. \end{aligned}$$

So, the L-parameter of the L-packet $\mathrm{sym}^3(\pi_\infty)$ is given by

$$re^{i\theta} \mapsto \begin{bmatrix} e^{i(k-1)\theta} &&& \\ & e^{i3(k-1)\theta} && \\ && e^{-i(k-1)\theta} & \\ &&& e^{-i3(k-1)\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} && (-1)^{3(k-1)} & \\ &&& (-1)^{3(k-1)} \\ 1 &&& \\ & 1 && \end{bmatrix}. \tag{3.9}$$

One can easily see from (2.48) that (3.9) is the L-parameter of a discrete series representation of $\mathrm{GSp}(4, \mathbb{R})$. So, the L-packet $\mathrm{sym}^3(\pi_\infty)$ contains a holomorphic discrete series representation with the Harish-Chandra parameter (λ_1, λ_2) such that $\lambda_1 + \lambda_2 = 3(k-1)$ and $\lambda_1 - \lambda_2 = (k-1)$, i.e., the Harish-Chandra parameter is $(2k-2, k-1)$. Then, using Table (44) in [42], the L-packet $\mathrm{sym}^3(\pi_\infty)$ has a holomorphic discrete series representation of $\mathrm{GSp}(4, \mathbb{R})$ with the minimal K-type $(\lambda_1 + 1, \lambda_2 + 2) = (2k-1, k+1)$. Using the notation introduced in Section 2.3.3, this is $\mathcal{B}_{k+1,k-2}$. It also has a large (or generic) discrete series representation with the Harish-Chandra parameter $(2k-2, -k+1)$.

3.5 Siegel modular forms coming via the symmetric cube lifting

We are interested in a classical version of Theorem 3.2.1 which associates a Siegel cusp form to a non-CM cuspidal newform via the sym^3 lifting. Moreover, we want to understand the level of the Siegel modular forms obtained by this lifting with respect to different congruence subgroups.

3.5.1 Automorphic representations and Siegel modular forms

There is a well-understood procedure of associating automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ with Siegel modular forms of degree 2. For a detailed description of this procedure we refer to [3] and Section 3.2 of [42]. Here we would like to mention some highlights of this connection.

Let $\Pi \cong \bigotimes_p \Pi_p$ be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, where Π_p is a representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$. Let V_p be a model for Π_p, so that $V \cong \otimes_p V_p$, a restricted tensor product. In order to get a holomorphic vector valued Siegel modular form, we need to make an assumption that Π_∞ is isomorphic to the lowest weight representation $\mathcal{B}_{k,j}$ of weight (k, j) defined in Section 2.3.3. Then Π_∞ contains the K-type $V(k + j, k)$ with multiplicity one. Let $v_\infty \in V_\infty$ be a non-zero vector of weight $(k + j, k)$ in this K-type. For each finite prime p, let v_p be a non-zero vector in V_p and C_p be an open-compact subgroup of $\mathrm{GSp}(4, \mathbb{Q}_p)$ stabilizing v_p. For almost all p, we may assume that v_p is the distinguished unramified vector and $C_p = \mathrm{GSp}(4, \mathbb{Z}_p)$. So, only for finitely many primes p, we need to make a choice for v_p and C_p. We will assume that the multiplier maps $\mu : C_p \to \mathbb{Z}_p^\times$ are surjective for all p. In this case, strong approximation for $\mathrm{Sp}(4)$ implies that $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q}) = \mathrm{GSp}(4, \mathbb{Q})\mathrm{GSp}(4, \mathbb{R})^+ \prod\limits_{p<\infty} C_p$. Also, by our choices, $\otimes v_p$ is a legitimate element in $\otimes_p V_p$ and it corresponds to a cusp form $\Phi \in V$ via the isomorphism $V \cong \otimes_p V_p$. Using the strong approximation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ and properties of the automorphic form Φ, we can find a vector valued Siegel cusp form f of weight (k, j) and degree 2 with respect to the following congruence subgroup of $\mathrm{Sp}(4, \mathbb{Q})$

$$\mathit{\Gamma} = \mathrm{GSp}(4, \mathbb{Q}) \cap \mathrm{GSp}(4, \mathbb{R})^+ \prod_{p<\infty} C_p, \tag{3.10}$$

which is associated to the given cuspidal automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ such that $L(s, f) = L(s, \Pi)$. Here $L(s, f)$ is the completed spin L-function of f defined in Section 3.3 of [42] and $L(s, \Pi) = \prod_{p\leq\infty} L(s, \Pi_p)$ is the completed spin L-function of Π. The spin L-factors $L(s, \Pi_p)$ of Π_p for finite primes p are listed in [37] and the spin L-factors $L(s, \Pi_\infty)$ of Π_∞ is given in (2.49). As in section 2.7.1, here also we consider $L(s, f)$ in the analytic normalization.

In the next subsections, we investigate the Siegel modular forms attached to $\Pi = \mathrm{sym}^3(\pi)$ with respect to different congruence subgroups Γ of $\mathrm{Sp}(4, \mathbb{Q})$ by making different choices for C_p. We consider the compact open subgroups $K(p^n)$, $\Gamma_0(p^n)$ and $\Gamma(p^n)$ of $\mathrm{GSp}(4, \mathbb{Q}_p)$ defined in (2.5), (2.6) and (2.7) respectively. For these three choices of C_p, we get Siegel modular forms with respect to the congruence subgroups $K(M)$, $\Gamma_0(M)$ and $\Gamma(M)$ of $\mathrm{Sp}(4, \mathbb{Q})$ as defined in Section 2.7.2. These are classical versions of the subgroups C_p with respect to the symplectic form $J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$.

3.5.2 Level with respect to the principal congruence subgroup

Let $\pi \cong \otimes_{p\leq\infty} \pi_p$ and $\Pi = \mathrm{sym}^3(\pi) \cong \otimes_{p\leq\infty} \mathrm{sym}^3(\pi_p)$ as in Theorem 3.2.1. Ramakrishnan and Shahidi considered the principal congruence subgroup level in [35]. They have mentioned that the finite component $\Pi_f \cong \otimes_{p<\infty} \Pi_p$ of $\Pi = \mathrm{sym}^3(\pi)$ has a non-zero vector invariant under the principal congruence subgroup of level equal to the conductor $a(\mathrm{sym}^3(\pi))$. Which means that for each $p \mid a(\pi)$, there exists a non-zero vector invariant under the principal congruence subgroup $\Gamma(p^{a(\mathrm{sym}^3(\pi_p))})$. But, the conductor $a(\mathrm{sym}^3(\pi_p)$ of $\mathrm{sym}^3(\pi_p)$ is *not* the minimal principal congruence level. In this subsection, we will describe the minimal principal congruence level of $\mathrm{sym}^3(\pi_p)$ when $\mathrm{sym}^3(\pi_p)$ is a non-supercuspidal representation.

First, we review the definition of depth of a representation defined by Moy-Prasad in [30]. Let G be a connected reductive algebraic group defined over a non-archimedean local field K. Let (π', V) be an irreducible admissible representation of G. Let $\mathcal{B} = \mathcal{B}(G)$ be the Bruhat-Tits building associated with G. Let $G_x = \{g \in G : g \cdot x = x\}$ be the parahoric subgroup of G associated to a point $x \in \mathcal{B}$. Moy and Prasad have defined a decreasing filtration of subgroups of G_x denoted by $G_{x,r}$ and indexed by the non-negative real number r. Let $G^+_{x,r} := \cup_{s>r} G_{x,s}$. Then Moy and Prasad showed that there exists a smallest rational number $r = \rho(\pi')$ such that the space $V^{G^+_{x,r}}$ is non-trivial for some $x \in \mathcal{B}$ and they called it the **depth** of π'. For general references for this paragraph, see [30] and [31].

Let $\rho(\mathrm{sym}^3(\pi_p))$ be the depth of the representation $\mathrm{sym}^3(\pi_p)$ of $\mathrm{GSp}(4, \mathbb{Q}_p)$. When the representation $\mathrm{sym}^3(\pi_p)$ of $\mathrm{GSp}(4, \mathbb{Q}_p)$ is non-supercuspidal, using Theorem 5.2 in [31], and the fact that for a character χ, the depth $\rho(\chi) = \max\{a(\chi) - 1, 0\}$, we list $\rho(\mathrm{sym}^3(\pi_p))$ in Table 3.3.

Fact 3.5.1. *If π_p is an essentially square integrable representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ then $\rho(\pi_p) = \max\left\{\frac{a(\pi_p)-2}{2}, 0\right\}$. This is a special case of a more general theorem in [28].*

Now, we note the following from Table 3.2 and Table 3.3 using Fact 3.5.1.

1. If $\mathrm{sym}^3(\pi_p) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$. Then $a(\mathrm{sym}^3(\pi_p)) = 2a(\chi^3) + 2a(\chi)$ and $\rho(\mathrm{sym}^3(\pi_p)) + 1 = \max\{a(\chi^2), a(\chi^4), a(\chi^3), 1\}$.

2. If $\mathrm{sym}^3(\pi_p) = \chi^3 \mathrm{St}_{\mathrm{GSp}(4)}$, then $a(\mathrm{sym}^3(\pi_p)) = 4a(\chi^3)$ and $\rho(\mathrm{sym}^3(\pi_p)) + 1 = \max\{a(\chi^3), 1\}$.

Table 3.3: Depth $\rho(\mathrm{sym}^3(\pi_p))$ of $\mathrm{sym}^3(\pi_p)$.

Here $\mathrm{sym}^3(\pi_p)$ is non-supercuspidal. $F/\mathbb{Q}_p$ is the unramified quadratic extension and ξ is the character of $F^\times$. Here, φ is a character of $\mathbb{Q}_p^\times$ such that $\xi^3 = \varphi \circ N_{F/\mathbb{Q}_p}$.

$\mathrm{sym}^3(\pi_p)$	Condition on π_p	$\rho(\mathrm{sym}^3(\pi_p))$
$\chi^4 \times \chi^2 \rtimes \chi^{-3}$		$\max\{a(\chi^2)-1, a(\chi^4)-1, a(\chi^3)-1, 0\}$
$\chi^3\mathrm{St}_{\mathrm{GSp}(4)}$	χ is ram.	$\max\{a(\chi^3)-1, 0\}$
	χ is unr.	0
$\omega_{F,\xi^3} \oplus \omega_{F,\xi}$	$\xi^4 = 1$ on $F^\times$	$\rho(\omega_{F,\xi})$
	$\xi^6 = 1$ on $F^\times$	$\max\{\rho(\varphi), \rho(\omega_{F,\xi^4})\}$

3. Let $\mathrm{sym}^3(\pi_p) = \omega_{F,\xi^3} \oplus \omega_{F,\xi}$ such that $F/\mathbb{Q}_p$ is a quadratic extension and ξ is a character of $F^\times$ with $\xi^\sigma = \xi^{-1}$. Let $\xi^4 = 1$ on $F^\times$. By Table 3.1, $\mathrm{sym}^3(\pi_p)$ is of type VIII. Then we have, $a(\mathrm{sym}^3(\pi_p)) = 2a(\omega_{F,\xi})$ and using Fact 3.5.1,

$$\rho(\mathrm{sym}^3(\pi_p)) + 1 = \max\left\{\frac{a(\omega_{F,\xi})-2}{2} + 1, 1\right\} = \max\left\{\frac{a(\omega_{F,\xi})}{2}, 1\right\}.$$

4. Let $\mathrm{sym}^3(\pi_p) = \omega_{F,\xi^3} \oplus \omega_{F,\xi}$ such that $F/\mathbb{Q}_p$ is a quadratic extension and ξ is a character of $F^\times$ with $\xi^\sigma = \xi^{-1}$. Let $\xi^6 = 1$ on $F^\times$. By Table 3.1, $\mathrm{sym}^3(\pi_p)$ is of type X and $\mathrm{sym}^3(\pi_p) = \omega_{F,\xi^4} \rtimes \varphi$, where φ is a character of $\mathbb{Q}_p^\times$ such that $\xi^3 = \varphi \circ N_{F/\mathbb{Q}_p}$. Then, $a(\mathrm{sym}^3(\pi_p)) = a(\omega_{F,\xi}) + 2a(\varphi)$ and using Fact 3.5.1,

$$\rho(\mathrm{sym}^3(\pi_p)) + 1 = \max\left\{\frac{a\left(\omega_{F,\xi^4}\right)}{2}, a(\varphi), 1\right\}.$$

Considering all the cases above, we see that $\rho(\mathrm{sym}^3(\pi_p))$ is an integer and

$$\rho(\mathrm{sym}^3(\pi_p)) + 1 < a(\mathrm{sym}^3(\pi_p)), \tag{3.11}$$

when $\mathrm{sym}^3(\pi_p)$ is a non-supercuspidal representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$.

Fact 3.5.2. *Using Theorem 5.2 of [30] for $G = \mathrm{GSp}(4)$, one can check that for an irreducible admissible representation (π', V) of $\mathrm{GSp}(4, \mathbb{Q}_p)$, if $\rho(\pi')$ is an integer then $n = \rho(\pi') + 1$ is the* smallest *integer n such that π' admits a non-zero vector under the principal congruence subgroup $\Gamma(p^n)$.*

Then, using Fact 3.5.2 and (3.11), we get the following remark.

Remark 3.5.3. *Let $\pi \cong \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ defined by a holomorphic, non-CM newform ϕ of even weight $k \geq 2$ and level N with trivial central character. Then there exists a cuspidal automorphic representation $\Pi \cong \bigotimes_p \Pi_p$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character, which is unramified at any prime p not dividing N, such that Π satisfies (i),(ii), (iii) of Theorem 3.2.1. If Π_p is a non-supercuspidal representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$ for each prime $p \mid N$, then the minimal principal congruence subgroup of level of Π is $\rho(\mathrm{sym}^3(\pi)) + 1$, i.e., the finite component $\Pi_f \cong \otimes_{p<\infty} \Pi_p$ of Π has a non-zero vector invariant under principal congruence subgroup of minimal level equal to $\rho(\mathrm{sym}^3(\pi)) + 1$. Here, $\rho(\mathrm{sym}^3(\pi))$ is the depth of $\mathrm{sym}^3(\pi)$.*

But, the principal congruence subgroups are not ideally suited for a correspondence between cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ and Siegel modular forms. So, we will not consider the principal congruence subgroup further.

3.5.3 Level with respect to the Siegel congruence subgroup

As before, assume that $\pi \cong \otimes_{p\leq\infty} \pi_p$ and $\Pi = \mathrm{sym}^3(\pi) = \otimes_{p\leq\infty} \mathrm{sym}^3(\pi_p)$ are as in Theorem 3.2.1. Recall from Section 3.5.1 that in order to find the Siegel congruence subgroup levels of the Siegel modular forms coming from the sym^3 lifting, one needs choose suitable vectors v_p invariant under the Siegel congruence subgroup $\Gamma_0(p^n)$ at each prime $p \mid a(\pi)$. So, one should study the space $V_0(n)$ of $\Gamma_0(p^n)$ invariant vectors of $\Pi_p = \mathrm{sym}^3(\pi_p)$.

First, we consider the space $V_0(2)$ for $\Pi_p = \mathrm{sym}^3(\pi_p)$ where π_p is a non-supercuspidal representation of $\mathrm{GL}(2, \mathbb{Q}_p)$. It turns out that the space $V_0(2)$ is very large, which is not ideal for choosing a good candidate $v_p \in V_0(2)$ to construct a holomorphic Siegel modular form. One of our sample results is the following when π_p is a principal series representation of $\mathrm{GL}(2, \mathbb{Q}_p)$.

Proposition 3.5.4. *Let $\pi_p = \chi \times \chi^{-1}$ with $a(\chi) = 1$, where χ is a character of $\mathbb{Q}_p^\times$. So, $\Pi_p = \mathrm{sym}^3(\pi_p) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$. Let $V_0(2)$ be the space of the $\Gamma_0(p^2)$ invariant vectors for the representation $\chi^4 \times \chi^2 \rtimes \chi^{-3}$. Then*

$$\textit{the dimension of } V_0(2) = \begin{cases} 6 & \textit{if } \chi^2 \textit{ is unramified,} \\ 3 & \textit{if } \chi^3 \textit{ is unramified,} \\ 4 & \textit{if } \chi^4 \textit{ is unramified and } \chi^2 \textit{ is ramified,} \\ 0 & \textit{otherwise.} \end{cases} \tag{3.12}$$

We omit the proof of this result since it is technical and not useful for our study of Siegel modular forms coming from the sym^3 lifting. Note that, $V_0(2)$ has a basis consisting of $f_g : \mathrm{GL}(2, \mathbb{Q}_p) \to \mathbb{C}^\times$, where g is a representative of the double coset $B \backslash g / \Gamma_0(p^2)$, defined as follows

$$f_g(x) = \begin{cases} \mid a^2 b \mid\mid c \mid^{-\frac{3}{2}} \chi^4(a)\chi^2(b)\chi^{-3}(c) & \text{if } x = \begin{bmatrix} a & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix} gk \in B \cdot g \cdot \Gamma_0(p^2), \\ 0 & \text{otherwise.} \end{cases}$$

Now the the *Atkin-Lehner* element $\eta = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^2 & & & \\ & -\varpi^2 & & \end{bmatrix}$ of level p^2 in $\mathrm{GSp}(4, \mathbb{Q}_p)$ acts on the space $V_0(2)$ as $\Pi_p(\eta)(f_g)(x) = f_g(x\eta)$. This gives us an operator $\eta : V_0(2) \to V_0(2)$, known as the Atkin-Lehner operator. Also, there is an endomorphism $\mu : V_0(2) \to V_0(2)$, known as the μ-operator, defined by

$$\Pi_p(\mu)(f_g) = \sum_{x,z \in \mathbb{Z}_p/p\mathbb{Z}_p} \Pi_p\left(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & z\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) f_g + \sum_{z \in \mathbb{Z}_p/p\mathbb{Z}_p} \Pi_p\left(\begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) f_g.$$

To find a suitable $\Gamma_0(p^2)$-invariant vector for $\mathrm{sym}^3(\pi_p) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$, we have also considered these two operators on the space $V_0(2)$ and computed the eigenvalues for them. But in most of the cases, we did not find a unique eigenvector which can be used for our purpose. As a conclusion of this subsection, we want to mention that the Siegel congruence subgroup is not a suitable congruence subgroup to study the Siegel modular forms coming from the sym^3 lifting.

3.5.4 Level with respect to the paramodular group

In this subsection, we consider Siegel modular forms with respect to the paramodular group. There is a well understood connection between paramodular forms and cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, and there is a nice newform theory for paramodular forms (see [36, 37]). These facts were not available at the time Ramakrishnan and Shahidi proved the result on the sym^3 lifting. Now, using Theorem 3.2.1 and the paramodular newform theory we get the following result.

Corollary 3.5.5. *Let ϕ be a non-CM cuspidal newform of even weight $k \geq 2$ and level N with trivial central character. Let $\pi \cong \bigotimes_p \pi_p$ be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ associated to ϕ. Then there exists a vector valued cuspidal paramodular newform f of weight $\det^{k+1} \times \mathrm{sym}^{k-2}$ and level equal to the conductor $a(\mathrm{sym}^3(\pi))$ of $\mathrm{sym}^3(\pi)$ such that $L(s, f) = L(s, \phi, \mathrm{sym}^3)$.*

Proof. By Theorem 3.2.1, π lifts to a cuspidal automorphic representation $\Pi = \mathrm{sym}^3(\pi)$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character. Moreover, Π is a representation of type **(G)** (see [44]) such that each non-archimedean component Π_p is generic and the archimedean component Π_∞ is a holomorphic discrete series representation. Then, by Theorem 4.4.1 from [37], there exists a paramodular vector of minimal level at each non-archimedean place. Now, recall from Theorem 7.5.4 and Corollary 7.5.5 of [37] that the minimal paramodular level of Π_p at each finite place p is the conductor $a(\Pi_p)$ of Π_p, and the dimension of the space of paramodular vectors at the minimal level is 1. Since $a(\Pi) = \prod_p p^{a(\Pi_p)}$, we get a paramodular newform f of level $a(\Pi) = a(\mathrm{sym}^3(\pi))$ associated to the automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ such that $L(s, f) = L(s, \Pi)$ (see [42]). Using part (ii) of Theorem 3.2.1 we get $L(s, f) = L(s, \Pi) = L(s, \pi, \mathrm{sym}^3) = L(s, \phi, \mathrm{sym}^3)$.

Also, one can easily see that the weight of f is $\det^{k+1} \times \mathrm{sym}^{k-2}$ by looking at the local parameter at the archimedean place as in (3.9) and the last paragraph of Section 3.4. This concludes the proof. $\square$

Remark 3.5.6. *In order to construct holomorphic Siegel modular forms, we choose the non-archimedean components of* $\mathrm{sym}^3(\pi)$ *to be generic and the archimedean components of* $\mathrm{sym}^3(\pi)$ *to be holomorphic discrete series in Theorem 3.2.1 and Corollary 3.5.5. In general, we consider* $\mathrm{sym}^3(\pi_p)$ *as the L-packet on* $\mathrm{GSp}(4, \mathbb{Q}_p)$ *whose L-parameter is the symmetric cube of the L-parameter of* π_p*. We can switch between generic and non-generic in each component since* $\mathrm{sym}^3(\pi)$ *is of type* **(G)***. But, when we study global results, we make the specific choice of local representations in the L-packet as in Theorem 3.2.1.*

Figure 3.1 shows the liftings in Theorem 3.2.1 and Corollary 3.5.5. We want to find *a formula for the level M* of the paramodular form f in terms of the data of the given newform ϕ. Since $M = a(\mathrm{sym}^3(\pi)) = \prod_p p^{a(\mathrm{sym}^3(\pi_p))}$, we need to calculate $a(\mathrm{sym}^3(\pi_p))$ for each local representation π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ attached to ϕ. Table 3.2 gives a formula for the conductor $a(\mathrm{sym}^3(\pi_p)$ of $\mathrm{sym}^3(\pi_p)$ in terms of the conductor of some characters that appear in $\mathrm{sym}^3(\pi_p)$.

$$\begin{tikzcd} \phi \in S_k^{\mathrm{new}}(\Gamma_0(N)) \arrow[rr, "\text{The sym}^3\text{ lifting}"] \arrow[d] & & F \in S^{\mathrm{new}}_{k+1,k-2}(K(M)) \\ \begin{array}{c}\pi \cong \bigotimes_p \pi_p \\ \text{on } \ \mathrm{GL}(2,\mathbb{A})\end{array} \arrow[r, "\text{Functoriality for sym}^3", "\text{(Kim-Shahidi)}"'] & \begin{array}{c}\mu = \bigotimes_p \mu_p \\ \text{on } \ \mathrm{GL}(4,\mathbb{A})\end{array} \arrow[r, "\text{(using Arthur packets)}", "\text{(Ramakrishnan-Shahidi)}"'] & \begin{array}{c}\Pi \cong \bigotimes_p \Pi_p \\ \text{on } \ \mathrm{GSp}(4,\mathbb{A})\end{array} \arrow[u] \end{tikzcd}$$

Figure 3.1: Diagram illustrating the symmetric cube lifting.

Notice that we need a description of the character involved in the L-parameter of π_p in order to calculate $a(\mathrm{sym}^3(\pi_p))$. So, the global question of finding the paramodular level coming from the sym^3 lifting leads to the local question of determining the representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ associated to a cusp form. In principle the paper [29] contains an algorithm to determine the local representations attached to modular forms. However, the local parameter of the output of the algorithm is not always obvious. So, we consider a more specified problem, namely finding the level of the paramodular forms coming from elliptic curves over $\mathbb{Q}$ through the sym^3 lifting. For that, we need to find a detailed description of the local representation π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ from an elliptic curve E over $\mathbb{Q}_p$. We discuss this in the next chapter.

Chapter 4

Representations attached to elliptic curves over a p-adic field

In this chapter we describe the representations associated to elliptic curves over a p-adic field. For elliptic curves with potential multiplicative reduction, one can find the associated representations for any p-adic field. If an elliptic curve has additive but potential good reduction over a p-adic field with residual characteristic ≥ 5, the associated representation is also known. The main focus of this chapter is to find a detailed description of the representations attached to elliptic curves with additive but potential good reduction over a non-archimedean local field of residual characteristic 3.

Let K be a non-archimedean local field of characteristic zero with residual characteristic p. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Let ϖ_K be a uniformizer of K. Suppose that $v : K \to \mathbb{Z}$ is the normalized valuation on K, and $k = \mathfrak{o}_K/\mathfrak{p}$ is the residue field of K of order q. Let $\Phi \in \mathrm{Gal}(\bar{K}/K)$ be an inverse Frobenius element, K^{un} be the maximal unramified extension of K inside $\bar{K}$, and I_K be the inertia group. Let E be an elliptic curve over K given by a minimal Weierstrass equation of the form $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$ as in (2.71). The discriminant Δ, the j-invariant $j(E)$, and the constants c_4, c_6 are the usual quantities attached to the Weierstrass equation given as in (2.72).

Let σ'_E be the Weil-Deligne representation associated to E/K as described in Section 2.8.4. Now, σ'_E corresponds to an irreducible admissible representation π_E of

$\mathrm{GL}(2,K)$ by the local Langlands correspondence. Here we give a description of π_E when the given elliptic curve E/K has bad reduction. We separate our study into a few sections depending on the reduction type of E.

4.1 Potential multiplicative reduction

If we assume E/K has potential multiplicative reduction, then c_4 and c_6 are non-zero since $j(E) \notin \mathfrak{o}_K$. Then we define the γ-invariant of E/K by

$$\gamma(E/K) = -\frac{c_4}{c_6} \in K^\times/K^{\times 2}. \tag{4.1}$$

This quantity is well defined and independent of the choice of the Weierstrass equation. Let σ'_E be the Weil-Deligne representation associated to E (see sections 13-15 in [39]), which corresponds to an irreducible admissible representation π_E of $\mathrm{GL}(2,K)$ by the local Langlands correspondence. We want a description of the representation π_E in terms of the Weierstrass coefficients of E. When E has potential multiplicative reduction, we have the following result from [39].

Theorem 4.1.1. *Let K be any non-archimedean local field of characteristic zero. Let E/K be an elliptic curve with potential multiplicative reduction, i.e., $j(E) \notin \mathfrak{o}_K$. Then the $\mathrm{GL}(2,K)$ representation π_E associated to E is given by*

$$\pi_E = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GL}(2,K)},$$

where $(\gamma(E/K), \cdot)$ is the quadratic character of $K^\times$ defined by the Hilbert symbol $(\cdot,\cdot)$. Furthermore, we have one of the following cases:

- *$(\gamma(E/K), \cdot)$ is trivial if and only if E/K has split multiplicative reduction.*

- $(\gamma(E/K), \cdot)$ *is non-trivial and unramified if and only if* E/K *has non-split multiplicative reduction.*

- $(\gamma(E/K), \cdot)$ *is ramified if and only if* E/K *has additive reduction.*

4.2 Potential good reduction in residual characteristic ≥ 5

When E has additive but potential good reduction and the residual characteristic of K is greater than or equal to 5, the following results from [50] describe the representation π_E.

Theorem 4.2.1. *Let* K *be a non-archimedean local field of characteristic zero with residual characteristic* ≥ 5*. Let* E/K *be an elliptic curve with additive but potential good reduction, i.e.,* $j(E) \in \mathfrak{o}_K$*. Assume that* $(q-1)v(\Delta) \equiv 0 \bmod 12$ *and* $e = \frac{12}{\gcd(v(\Delta),12)}$*. Then the* $\mathrm{GL}(2, K)$ *representation* π_E *associated to* E *is a principal series representation, i.e.,* $\pi_E = \chi \times \chi^{-1}$*, where* χ *is a character of* $K^\times$ *satisfying the following properties:*

(i) The conductor $a(\chi)$ *of* χ *is 1, i.e.,* χ *is trivial on* $1 + \mathfrak{p}$*,*

(ii) $\chi|_{W_{q-1}}$ *is trivial on the index* e*-subgroup* W_{q-1}^e*, where* W_{q-1} *is the group of* $(q-1)$*th root of unity,*

(iii) The character of $W_{q-1}/W_{q-1}^e \cong \mathbb{Z}/e\mathbb{Z}$ *induced by* χ *has order* e*.*

Furthermore, there is a unique such character when $e = 2$*, and there are exactly two such characters when* $e \in \{3, 4, 6\}$ *which are inverses of each other.*

Theorem 4.2.2. *Let* K *be a non-archimedean local field of characteristic zero with residual characteristic* ≥ 5*. Let* E/K *be an elliptic curve with additive but potential good reduction, i.e.,* $j(E) \in \mathfrak{o}_K$*. Assume that* $(q-1)v(\Delta) \not\equiv 0 \bmod 12$ *and* $e = \frac{12}{\gcd(v(\Delta),12)}$*.*

Then the corresponding $\mathrm{GL}(2, K)$ *representation* π_E *is a dihedral supercuspidal representation with trivial central character, i.e.,* $\pi_E = \omega_{F,\xi}$*, where* F *is the unramified quadratic extension of* K *and* ξ *is a character of* $F^\times$ *satisfying the following properties:*

(i) The conductor $a(\xi)$ *of* ξ *is 1, i.e.,* ξ *is trivial on* $1 + \mathfrak{p}_F$*,*

(ii) $\xi|_{\mathfrak{o}_F^\times}$ *has order* $e \in \{3, 4, 6\}$*,*

(iii) $\xi(\varpi_F) = -1$*. Here,* ϖ_F *is a uniformizer of* F *chosen to be in* K*.*

Furthermore, there are exactly two such characters which are Galois conjugates of each other for each e*.*

4.3 Potential good reduction in residual characteristic 3 and $v(3) = 1$

Assume that the residual characteristic of K is 3 and $v(3) = 1$. In Table 4.1, we define a list of conditions on E in terms of the quantities $v(c_4)$, $v(c_6)$, and $v(\Delta)$. This table is reproduced from Table II of [33]. For $i = 2$ or 5, we define a condition R_i on E/K by

$$\mathrm{R}_i : \quad x^3 - 3c_4x - 2c_6 \equiv 0 \bmod (27\varpi_{\mathrm{K}}^{\mathrm{i}}) \text{ for some } \mathrm{x} \in \mathfrak{o}_{\mathrm{K}}. \tag{4.2}$$

If an elliptic curve E/K does not satisfy the condition R_i, then we denote it by "non R_i" in Table 4.1. When $K = \mathbb{Q}_3$, $v(c_4) \geq 2$, and $v(c_6) = 3$, the condition R_2 is equivalent to

$$(c_6/3^3)^2 + 2 \equiv c_4/3 \bmod 9. \tag{4.3}$$

When $K = \mathbb{Q}_3$, $v(c_4) \geq 4$, and $v(c_6) = 6$, the condition R_5 is equivalent to

$$(c_6/3^6)^2 + 2 \equiv c_4/3^3 \bmod 9. \tag{4.4}$$

There are also some conditions on the underlying field K in Table 4.1, called "reducibility condition", that determines whether the Galois representation attached to E is reducible or irreducible. The exponent $v(N)$ of the conductor N of the elliptic curve E appears on the last column of Table 4.1. The list of conditions does not depend on $v(N)$. In fact $v(N)$ can be determined in terms of the quantities $v(c_4)$, $v(c_6)$, and $v(\Delta)$. For each condition in Table 4.1, we describe the $\mathrm{GL}(2, K)$ representation π_E associated to E. The following lemma ensures that this way we get all the possible $\mathrm{GL}(2, K)$ representations associated to an elliptic curve E/K with additive but potential good reduction.

Lemma 4.3.1. *Let K be a non-archimedean local field of characteristic zero with residual characteristic 3 and $v(3) = 1$. Suppose that E is an elliptic curve over K given by a minimal Weierstrass equation of the form (2.71) with the coefficients in $\mathfrak{o}_K$. Let Δ be the discriminant, and c_4, c_6 be the usual constants attached to the equation (2.71) as defined in (2.72). Then the following statements are equivalent:*

(i) E has additive but potential good reduction.

(ii) E satisfies one and only one of the conditions in Table 4.1.

Proof. The Kodaira-Néron type of E/K is one of the types described in Theorem 8.2 of [48], and it can be determined in terms of the coefficients of (2.71) using Tate's algorithm (see our discussion in Section 2.8.5). Now, the proof follows from the following equivalent steps:

E has additive but potential good reduction.

$\Leftrightarrow$ The j-invariant $j(E)$ is integral, i.e., $v\left(j(E)\right) \geq 0$. (See Proposition 5.5 in [47].)

$\Leftrightarrow$ $3v\left(c_4\right) \geq v(\Delta)$. (By definition of $j(E)$ as in (2.72).)

$\Leftrightarrow$ The possible Kodaira-Néron types of E are $\mathrm{I}_0^*, \mathrm{II}, \mathrm{II}^*, \mathrm{III}, \mathrm{III}^*, \mathrm{IV}, \mathrm{IV}^*$.

(By Tate's algorithm. Also see Table II in [33].)

$\Leftrightarrow$ E satisfies one and only one of the conditions in Table 4.1. (By Table II in [33].) $\square$

Assume that E/K has additive but potential good reduction, i.e., E satisfies one of the conditions in Table 4.1. Let L/K^{un} be the smallest extension such that E/L has good reduction. By the corollary after Lemme 3 of [26], we have

$$L = K^{\mathrm{un}}(E[2], \Delta^{\frac{1}{4}}), \tag{4.5}$$

where $K^{\mathrm{un}}(E[2])$ is the splitting field of the polynomial on the right hand side of (2.71). Now the kernel of $\sigma'_E : W(\bar{K}/K) \to \mathrm{GL}(2, \mathbb{C})$ is $\mathrm{Gal}(\bar{K}/L)$ (see section 2 of [38]). So, $\sigma'_E : W(L/K) \cong W(\bar{K}/K)/\mathrm{Gal}(\bar{K}/L) \to \mathrm{GL}(2, \mathbb{C})$ is a faithful representation. Let $\Lambda = \mathrm{Gal}(L/K^{\mathrm{un}})$. Then, we have

$$I_K/\mathrm{Gal}(\bar{K}/L) \cong \Lambda \text{ and } W(L/K) = \Lambda \rtimes \langle \Phi \rangle\,, \tag{4.6}$$

and

$$\sigma'_E(I_K) = \sigma'_E(I_K/\mathrm{Gal}(\bar{K}/L)) = \sigma'_E(\Lambda) \cong \Lambda. \tag{4.7}$$

Remark 4.3.2. *Since σ'_E is a faithful representation of $W(L/K)$, using (4.6) and (4.7) we conclude that the order of $\sigma'_E|_{I_K}$ is $|\Lambda|$.*

Also, we use the following remark in the proofs of the main theorems of this section.

Remark 4.3.3. *A faithful two-dimensional semisimple complex representation of a group is reducible if and only if the group is abelian. So, the Weil-Deligne representation σ'_E associated to the given elliptic curve E/K is reducible if and only if $W(L/K)$ is abelian, i.e., the image of σ'_E is abelian (also see Proposition 2 in [38]).*

4.3.1 Principal series representations

The following theorem describes all possible principal series representations of $\mathrm{GL}(2, K)$ coming from elliptic curves over K with additive but potential good reduction.

Table 4.1: Table of conditions in terms of the quantities $v(\Delta)$, $v(c_4)$, and $v(c_6)$.

Name of condition	Reducibility condition	$v(\Delta)$	$v(c_4)$	$v(c_6)$	Additional condition	Néron type	$v(N)$
P_2		6	2	3		I_0^*	2
			3	≥ 6			
P_4	$-1 \in k^{\times 2}$	3	≥ 2	3	R_2 as in (4.2)	III	2
			2	≥ 5			
		9	≥ 4	6	R_5 as in (4.2)	III^*	
			4	≥ 8			
S_4	$-1 \notin k^{\times 2}$	3	≥ 2	3	R_2 as in (4.2)	III	2
			2	≥ 5			
		9	≥ 4	6	R_5 as in (4.2)	III^*	
			4	≥ 8			
P_3	$\Delta \in K^{\times 2}$	4	2	3		II	4
		12	5	8		II^*	
S_3	$\Delta \notin K^{\times 2}$	4	2	3		II	4
		12	5	8		II^*	
P_6	$\Delta \in K^{\times 2}$	6	3	5		IV	4
		10	4	6		IV^*	
S_6	$\Delta \notin K^{\times 2}$	6	3	5		IV	4
		10	4	6		IV^*	
S_6'	$\Delta \notin K^{\times 2}$	3	≥ 2	3	non R_2	II	3
			2	4			
		5	2	3		IV	
		9	≥ 4	6	non R_5	IV^*	
			4	7			
		11	4	6		II^*	
S_6''	$\Delta \notin K^{\times 2}$	5	≥ 3	4		II	5
		7	≥ 4	5		IV	
		11	≥ 5	7		IV^*	
		13	≥ 6	8		II^*	

Theorem 4.3.4. *Let K be a non-archimedean local field of characteristic zero with residual characteristic 3 and $v(3) = 1$. Let E/K be an elliptic curve given by a minimal Weierstrass equation of the form (2.71) with the coefficients in $\mathfrak{o}_K$. Assume that E satisfies one of the conditions in $\{\mathrm{P}_2, \mathrm{P}_4, \mathrm{P}_3, \mathrm{P}_6\}$ as defined in Table 4.1. Then the corresponding $\mathrm{GL}(2, K)$ representation π_E is a principal series representation, i.e., $\pi_E = \chi \times \chi^{-1}$, where χ is a character of $K^\times$ satisfying the following properties:*

(i) If E satisfies the condition P_m, then $\chi|_{\mathfrak{o}_K^\times}$ has order m. Here, $m \in \{2, 3, 4, 6\}$.

(ii) The conductor $a(\chi)$ of χ is given by

$$a(\chi) = \begin{cases} 1 & \textit{if } E \textit{ satisfies } \mathrm{P}_2 \textit{ or } \mathrm{P}_4, \\ 2 & \textit{if } E \textit{ satisfies } \mathrm{P}_3 \textit{ or } \mathrm{P}_6. \end{cases}$$

Furthermore, there is a unique such character on $\mathfrak{o}_K^\times$ when E satisfies P_2, and there are exactly two such characters on $\mathfrak{o}_K^\times$ when E satisfies P_4, which are inverses of each other.

Before proving Theorem 4.3.4, we review a few facts about the correspondence between the set of characters of $W(L/K)$ and the set of characters of $K^\times$ using Artin isomorphism. Here, we consider the following diagram.

$$\begin{tikzcd}
\bar{K} \arrow[d, dash] \arrow[dd, bend right=60, dash, "I_K"'] \\
L \arrow[d, dash] \arrow[d, bend left=60, dash, "\Lambda"] \\
K^{\mathrm{un}} \arrow[d, dash] \arrow[d, bend right=60, dash, "\langle \Phi \rangle"'] \\
K \arrow[d, dash] \\
\mathbb{Q}_3
\end{tikzcd} \tag{4.8}$$

Suppose that, $\sigma'_E = \chi \oplus \chi^{-1}$ where χ is a character of $W(L/K)$. By Remark 4.3.2, $\sigma'_E|_{I_K} = \sigma'_E|_\Lambda$ has order $|\Lambda|$. Then, using the proof of Théorème 1 of [26] (also see Theorem 3.1 of [24]), one can see that Λ is always cyclic if E satisfies one of conditions $\mathrm{P_i}$, $i \in \{1, 2, 3, 4\}$. So, $\sigma'_E(\Lambda)$ has order $|\Lambda|$ is the same as saying $n = |\Lambda|$ is the smallest integer n such that $\sigma'_E(w)^n = 1$ for all $w \in \Lambda$.

Fact 4.3.5. *$\chi|_\Lambda$ also has order $|\Lambda|$, i.e., $n = |\Lambda|$ is the smallest integer n such that $\chi^n = 1$ on Λ.*

Proof. We have the following matrix representation of σ'_E:

$$\sigma'_E(w) = \begin{bmatrix} \chi(w) & \\ & \chi^{-1}(w) \end{bmatrix}, \quad w \in W(L/K). \tag{4.9}$$

Let m be the order of $\chi|_\Lambda$. Now,

$$\chi(w)^{|\Lambda|} = \chi(w^{|\Lambda|}) = 1 \text{ for all } w \in \Lambda. \tag{4.10}$$

So, m divides $|\Lambda|$. If possible, let $m < |\Lambda|$. Since order of $\chi|_\Lambda$ = order of $\chi^{-1}|_\Lambda = m$,

$$\sigma'_E(w)^m = \begin{bmatrix} \chi(w) & \\ & \chi^{-1}(w) \end{bmatrix}^m = 1. \tag{4.11}$$

This is a contradiction since the order of $\sigma'_E|_\Lambda$ is $|\Lambda|$. Hence, Fact 4.3.5 is true. $\square$

Next, we find the character of $K^\times$ which corresponds to the character χ of $W(L/K)$

using the following two commutative diagrams:

$$\begin{tikzcd}
I_K \arrow[r] \arrow[d, hook] & I_K/(I_K \cap W(\bar{K}/L)) \cong \Lambda \arrow[d, hook] & \\
W(\bar{K}/K) \arrow[r, "p"] \arrow[rr, bend right, "\chi\circ p"'] & W(\bar{K}/K)/W(\bar{K}/L) \cong W(L/K) \arrow[r, "\chi"] & \mathbb{C}^\times
\end{tikzcd} \tag{4.12}$$

$$\begin{tikzcd}
\mathfrak{o}_K^\times \arrow[r, "\sim"] \arrow[d, hook] & \bar{p}(I_K) \arrow[d, hook] & I_K \arrow[l, "\bar{p}"'] \arrow[d, hook] \\
K^\times \arrow[r, "\sim", "r_K"'] \arrow[rrd, bend right, "\hat{\chi}"] & W(\bar{K}/K)^{\mathrm{ab}} \cong W(\bar{K}/K)/W(\bar{K}/K)^c \arrow[rd, "\bar{\chi}"] & W(\bar{K}/K) \arrow[l, "\bar{p}"'] \arrow[d, "\chi\circ p"] \\
 & & \mathbb{C}^\times
\end{tikzcd} \tag{4.13}$$

We start with the character $\chi : W(L/K) \to \mathbb{C}^\times$. Using Diagram (4.12), we get the character $\chi \circ p$ of $W(\bar{K}/K)$. Here, p is the quotient map from $W(\bar{K}/K)$ to $W(L/K)$. Then using Diagram (4.13), we get the character $\bar{\chi}$ of $W(\bar{K}/K)^{\mathrm{ab}}$ such that $\bar{\chi} \circ \bar{p} = \chi \circ p$. Here, $\bar{p}$ is the quotient map from $W(\bar{K}/K)$ to $W(\bar{K}/K)^{\mathrm{ab}}$. Then finally we get the character $\hat{\chi} = \bar{\chi} \circ r_K$ of $K^\times$ which corresponds to the character χ of $W(L/K)$. Here r_K is the Artin isomorphism. One can easily check that the following map

$$\begin{aligned} \{\text{Characters of } W(L/K)\} &\to \big\{\text{Characters of } K^\times\big\} \\ \chi &\mapsto \hat{\chi}. \end{aligned} \tag{4.14}$$

is a homomorphism of groups of characters such that we have the following remark.

Remark 4.3.6. *Assume that $\sigma'_E = \chi \oplus \chi^{-1}$ be the Weil-Deligne representation associated to E/K, where χ is a character of $W(L/K)$. By the local Langlands correspondence, $\pi_E = \hat{\chi} \times \hat{\chi}^{-1}$, where $\hat{\chi}$ is the character of $K^\times$ corresponds to χ via (4.14). Then by chasing the diagrams (4.12), (4.13), and using Fact 4.3.5 one can show that*

1. $\hat{\chi}^{-1} = \widehat{\chi^{-1}}$.

2. $\hat{\chi}|_{\mathfrak{o}_K^\times}$ *has order* $|\Lambda|$.

For simplicity of notation, we denote the character $\hat{\chi}$ *of* $K^\times$ *by* χ *too.*

Proof of Theorem 4.3.4. Suppose E satisfies the condition P_m for $m \in \{2, 3, 4, 6\}$. By Théorème 1 of [26] (also see Theorem 3.1 of [24]), we have $\Lambda \cong \mathbb{Z}/m\mathbb{Z}$. Now, Proposition 3.2 of [24] implies that $W(L/K)$ is abelian, i.e., the image of σ'_E is abelian. Then, using Remark 4.3.3 (also see Proposition 3.3 of [24]), we get $\sigma'_E \cong \chi \oplus \chi^{-1}$, where χ is a character of $W(L/K)$. By the local Langlands correspondence, the corresponding $\mathrm{GL}(2, K)$ representation π_E is the principal series representation $\chi \times \chi^{-1}$, where χ is the corresponding character of $K^\times$. Using Remark 4.3.6 we see that $\chi|_{\mathfrak{o}_K^\times}$ has order m. From Table 4.1, $a(\pi_E) = v(N) = 2$ when E satisfies either P_2 or P_4, which implies $a(\chi) = 1$. In this case, we get an induced character $\chi : \mathfrak{o}_K^\times/(1 + \mathfrak{p}) \cong k^\times \to \mathbb{C}^\times$. Note that $k^\times$ is a cyclic group of order $3^n - 1$, where K has degree n over $\mathbb{Q}_3$.

When E satisfies the condition P_2, the induced character $\chi : k^\times \to \mathbb{C}^\times$ has order 2 and $2 \mid 3^n - 1$. There is only one element of order 2 in $k^\times$. So, the induced character is the unique such character of order 2.

When E satisfies the condition P_4, the induced character $\chi : k^\times \to \mathbb{C}^\times$ has order 4. Since $-1 \in k^{\times 2}$, $k^\times$ contains the root of $x^2 + 1$. Then the quadratic extension $\mathbb{Q}_3(i)$ of $\mathbb{Q}_3$ is contained in K.

$$\begin{tikzcd}[arrows=dash] K \arrow[d] \\ \mathbb{Q}_3(i) \arrow[d, "2"] \\ \mathbb{Q}_3 \end{tikzcd}$$

So, $4 \mid 3^n - 1$. Now, there are exactly $\varphi(4) = 2$ elements of order 4 in $k^\times$. So, there are exactly two such characters χ on $\mathfrak{o}_K^\times$, which are inverses of each other.

Similarly, when E satisfies P_3 or P_6, $a(\pi_E) = v(N) = 4$ from Table 4.1. So, $a(\chi) = 2$

in these two cases. Hence, we get all the cases of Theorem 4.3.4. $\square$

Corollary 4.3.7. *Let E be an elliptic curve over $\mathbb{Q}_3$ given by a minimal Weierstrass equation of the form (2.71) with the coefficients in $\mathbb{Z}_3$. Assume that E satisfies one of the conditions in $\{\mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_6\}$ as defined in Table 4.1. Then the corresponding $\mathrm{GL}(2, \mathbb{Q}_3)$ representation is a principal series representation, i.e., $\pi_E = \chi \times \chi^{-1}$, where χ is a character $\mathbb{Q}_3^\times$ satisfying the following properties:*

(i) If E satisfies the condition P_m, then $\chi|_{\mathbb{Z}_3^\times}$ has order m with $m \in \{2, 3, 6\}$.

(ii) The conductor $a(\chi)$ of χ is given by

$$a(\chi) = \begin{cases} 1 & \textit{if } E \textit{ satisfies } \mathrm{P}_2, \\ 2 & \textit{if } E \textit{ satisfies } \mathrm{P}_3 \textit{ or } \mathrm{P}_6. \end{cases}$$

(iii) Furthermore, there is a unique such character on $\mathbb{Z}_3^\times$ when E satisfies P_2, and there are exactly two such characters on $\mathbb{Z}_3^\times$ when E satisfies P_3 or P_6, which are inverses of each other.

Proof. This corollary is a special case of Theorem 4.3.4 for $K = \mathbb{Q}_3$. So, most of the statements follow from the proof of Theorem 4.3.4 except for the property (iii) when E satisfies P_3 or P_6.

Note that when E satisfies P_3 or P_6, $a(\chi) = 2$. So, the character $\chi|_{\mathbb{Z}_3^\times}$ induces a character on $\mathbb{Z}_3^\times/(1 + 3^2\mathbb{Z}_3) \cong (\mathbb{Z}/9\mathbb{Z})^\times$. Since $(\mathbb{Z}/9\mathbb{Z})^\times$ is cyclic group of order 6, there are exactly $\varphi(3) = 2$ (resp. $\varphi(6) = 2$) elements of order 3 (resp. order 6) in $(\mathbb{Z}/9\mathbb{Z})^\times$ which are inverses of each other. Now, if E satisfies the condition P_3 (resp. P_6), then the character $\chi|_{\mathbb{Z}_3^\times}$ has order 3 (resp. order 6). Hence, there are exactly two such characters on $\mathbb{Z}_3^\times$ when E satisfies P_3 or P_6. This completes the proof of the corollary. $\square$

Note: $E/\mathbb{Q}_3$ *never* satisfies P_4 since $-1 \notin \mathbb{Q}_3^{\times 2}$.

4.3.2 Supercuspidal representations

Let G be a group, and H be an index-2 subgroup of G. All representations of these groups are assumed to be finite-dimensional and complex. Let $\sigma \in G \setminus H$, so that $G = H \cup \sigma H$. If ξ is a representation of H, then the conjugate representation ξ^σ is defined by $\xi^\sigma(h) = \xi(\sigma h \sigma^{-1})$. We denote the restriction functor by res_H^G and the induction functor by ind_H^G. Then we have the following result which we use to prove our main theorem of this subsection.

Lemma 4.3.8. *Let G be a group, and H be an index-2 subgroup of G. Let χ be the unique non-trivial character of G/H. Let φ be an irreducible representation of G. Then exactly one of the following holds*

1. *$\varphi \ncong \varphi \otimes \chi$ and $\mathrm{res}_H^G \varphi$ is irreducible. In that case $\mathrm{ind}_H^G(\mathrm{res}_H^G(\varphi)) = \varphi \oplus (\varphi \otimes \chi)$.*

2. *$\varphi \cong \varphi \otimes \chi$ and $\mathrm{res}_H^G \varphi = \xi \oplus \xi^\sigma$, where ξ is a representation of H. In that case $\xi \ncong \xi^\sigma$, and $\varphi = \mathrm{ind}_H^G(\xi) = \mathrm{ind}_H^G(\xi^\sigma)$.*

The following theorem describes all the supercuspidal representations of $\mathrm{GL}(2, K)$ associated to elliptic curves over K with additive but potential good reduction.

Theorem 4.3.9. *Let K be a non-archimedean local field of characteristic zero with residual characteristic 3 and $v(3) = 1$. Let E/K be an elliptic curve given by a minimal Weierstrass equation of the form (2.71) with the coefficients in $\mathfrak{o}_K$. Assume that E satisfies one of the conditions in $\{\mathrm{S}_4, \mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6', \mathrm{S}_6''\}$ as defined in Table 4.1. Then the associated $\mathrm{GL}(2, K)$ representation π_E is a dihedral supercuspidal representation, i.e., $\pi_E = \omega_{F,\xi}$, where F is a quadratic extension of K and ξ is a character of $F^\times$ satisfying the following properties:*

(i)

$$F = \begin{cases} K(i) & \textit{is the unramified extension of } K \textit{ if } E \textit{ satisfies } \mathrm{S}_4, \\ K(\sqrt{\Delta}) & \textit{is the unramified extension of } K \textit{ if } E \textit{ satisfies } \mathrm{S}_3 \textit{ or } \mathrm{S}_6, \\ K(\sqrt{\Delta}) & \textit{is a ramified extension of } K \textit{ if } E \textit{ satisfies } \mathrm{S}_6' \textit{ or } \mathrm{S}_6''. \end{cases}$$

(ii) The conductor $a(\xi)$ of ξ is given by

$$a(\xi) = \begin{cases} 1 & \textit{if } E \textit{ satisfies } \mathrm{S}_4, \\ 2 & \textit{if } E \textit{ satisfies a condition in } \left\{\mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6'\right\}, \\ 4 & \textit{if } E \textit{ satisfies } \mathrm{S}_6''. \end{cases}$$

(iii) If E satisfies the condition S_m for $m \in \{3,4,6\}$, then $\xi|_{\mathfrak{o}_F^\times}$ has order m. If E satisfies the condition S_6' or S_6'', then $\xi|_{\mathfrak{o}_F^\times}$ has order 6.

(iv) $\xi(\varpi_F) = -1$ when E satisfies one of the conditions in $\{\mathrm{S}_4, \mathrm{S}_3, \mathrm{S}_6\}$. Here, ϖ_F is a uniformizer of F chosen to be in K.

(v) In all cases, we have $\xi^\sigma = \xi^{-1}$ on $F^\times$.

Furthermore, when E satisfies S_4, there are exactly two such characters on $\mathfrak{o}_F^\times$ and they are Galois conjugates of each other.

Suppose that $\sigma_E' = \operatorname{ind}_{W(L/F)}^{W(L/K)}(\xi)$ where ξ is a character of $W(L/F)$. We know that σ_E' is a faithful representation of $W(L/K)$ with kernel $\mathrm{Gal}(\bar{K}/L)$. Then $\sigma_E'|_{I_F} = \sigma_E'|_{I_F/\mathrm{Gal}(\bar{K}/L)} = \sigma_E'|_{\mathrm{Gal}(L/F^{\mathrm{un}})}$ has order $|\mathrm{Gal}(L/F^{\mathrm{un}})|$. Here, we need to consider the following two different diagrams depending on F being the unramified or a ramified quadratic extension of K.

$$
\begin{tikzcd}[column sep=small]
\bar{K} = \bar{F} \arrow[d, dash] \arrow[dd, dash, bend right=50, "I_K = I_F"'] \\
L \arrow[d, dash] \arrow[d, dash, bend left=50, "\Lambda"] \\
K^{\mathrm{un}} = F^{\mathrm{un}} \arrow[d, dash] \arrow[d, dash, bend left=50, "\langle \Phi^2 \rangle"] \arrow[dd, dash, bend right=50, "\langle \Phi \rangle"'] \\
F \arrow[d, dash] \arrow[d, dash, bend left=50, "\langle \sigma \rangle"] \\
K
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bar{K} = \bar{F} \arrow[d, dash] \arrow[dd, dash, bend right=50, "I_F"'] & \\
L \arrow[d, dash] \arrow[dd, dash, bend left=60, "\Lambda"] & \\
F^{\mathrm{un}} \arrow[d, dash] \arrow[dr, dash] & \\
K^{\mathrm{un}} \arrow[d, dash] \arrow[d, dash, bend right=50, "\langle \Phi \rangle"'] & F \\
K \arrow[ur, dash] &
\end{tikzcd}
\tag{4.15}
$$

Fact 4.3.10. $\xi|_{\mathrm{Gal}(L/F^{\mathrm{un}})}$ *also has order* $|\mathrm{Gal}(L/F^{\mathrm{un}})|$.

One can prove this fact in a similar manner as we have proved Fact 4.3.5. Note that, when F is unramified, $\mathrm{Gal}(L/F^{\mathrm{un}}) = \mathrm{Gal}(L/K^{\mathrm{un}}) = \Lambda$. Before proving Theorem 4.3.9, we find the character of $F^\times$ which corresponds to the character ξ of $W(L/F)$ and show the relationship between them using the following two commutative diagrams:

$$
\begin{tikzcd}
I_F \arrow[r] \arrow[d, hook] & I_F/(I_F \cap W(\bar{K}/L)) \cong W(L/F^{\mathrm{un}}) \arrow[d, hook] & \\
W(\bar{K}/F) \arrow[r, "p"] \arrow[rr, bend right=15, "\xi \circ p"'] & W(\bar{K}/F)/W(\bar{K}/L) \cong W(L/F) \arrow[r, "\xi"] & \mathbb{C}^\times
\end{tikzcd}
\tag{4.16}
$$

$$
\begin{tikzcd}
\mathfrak{o}_F^\times \arrow[r] \arrow[d, hook] & \bar{p}(I_F) \arrow[d, hook] & I_F \arrow[l] \arrow[d, hook] \\
F^\times \arrow[r, "\cong", "r_F"'] \arrow[rrd, bend right=15, "\hat{\xi}"] & W(\bar{K}/F)^{\mathrm{ab}} \cong W(\bar{K}/F)/W(\bar{K}/F)^c \arrow[rd, "\bar{\xi}"] & W(\bar{K}/F) \arrow[l, "\bar{p}"'] \arrow[d, "\xi \circ p"] \\
 & & \mathbb{C}^\times
\end{tikzcd}
\tag{4.17}
$$

Using Diagrams (4.16), (4.17) and the Artin isomorphism r_F we get the character $\hat{\xi}$ of $F^\times$ which corresponds to the character ξ of $W(L/F)$ and we have the following remark.

Remark 4.3.11. *Suppose $\sigma'_E = \mathrm{ind}_{W(L/F)}^{W(L/K)}\xi$ is the Weil-Deligne representation associated to E/K, where ξ is a character of $W(L/F)$. By the local Langlands correspondence, $\pi_E = \omega_{F,\hat{\xi}}$ is a dihedral supercuspidal representation of $\mathrm{GL}(2,K)$, where $\hat{\xi}$ is the character of $F^\times$ corresponds to ξ as above. Then $\hat{\xi}$ satisfies the following properties*

(1) $\widehat{\xi^{-1}} = (\hat{\xi})^{-1}$

(2) $\widehat{\xi^{\sigma}} = (\hat{\xi})^{\sigma}$

(3) $\hat{\xi}|_{\mathfrak{o}_F^\times}$ *has order* $|\mathrm{Gal}(L/F^{\mathrm{un}})|$.

(4) The character

$$\det(\sigma'_E) : w \mapsto \xi(w)\xi^{\sigma}(w), \quad w \in W(L/F)$$
$$\sigma \mapsto -\xi(\sigma^2), \qquad \sigma \in W(L/K) \setminus W(L/F)$$

of $W(L/K)$ corresponds to the character $\hat{\xi}|_{K^\times} \cdot \chi_{F/K}$ of $K^\times$, where $\chi_{F/K}$ is the quadratic character of $K^\times$ associated to the quadratic extension F/K.

For simplicity of notation, we denote the character $\hat{\xi}$ of $F^\times$ by ξ too.

Proof of Theorem 4.3.9. We prove the theorem in two cases.

Case 1: Suppose that E satisfies the condition S_m for some $m \in \{4, 3, 6\}$. Then by Théorème 1 of [26] (also see Theorem 3.1 of [24]), we have $\Lambda \cong \mathbb{Z}/m\mathbb{Z}$. For this case we consider the diagram on the left of (4.15).

If E satisfies the condition S_4, then $-1 \notin k^{\times 2}$. By Hensel's lemma $-1 \in \mathfrak{o}_K^\times \setminus \mathfrak{o}_K^{\times 2}$. So, the field $F = K(i)$ is the unique unramified extension of K of degree 2. Now, $W(L/K) = \mathbb{Z}/4\mathbb{Z} \rtimes \langle\Phi\rangle$ and Φ^2 acts trivially on $\mathbb{Z}/4\mathbb{Z}$ (since the non-trivial action of Φ on $\mathbb{Z}/4\mathbb{Z}$ sends $1 \mapsto 3$, $3 \mapsto 1$, and fixes $0, 2$). So, we get $W(L/F) = \mathbb{Z}/4\mathbb{Z} \times \langle\Phi^2\rangle$. Note that σ'_E is a two dimensional faithful representation of $W(L/K)$. Since $-1 \notin k^{\times 2}$, by Proposition 3.2 of [24], $W(L/K)$ is not abelian. Then, by Remark 4.3.3 (also see Proposition 3.3 of [24]), σ'_E is an irreducible representation. Now, since $W(L/F)$ is abelian, by Remark 4.3.3 and

Lemma 4.3.8, $\mathrm{res}_{W(L/F)}^{W(L/K)}(\sigma'_E) = \xi \oplus \xi^\sigma$ for some character ξ of $W(L/F)$ with $\xi \neq \xi^\sigma$ and $\sigma'_E = \mathrm{ind}_{W(L/F)}^{W(L/K)}(\xi)$. Here σ is an element of $W(L/K) \setminus W(L/F)$. Hence, by the local Langlands correspondence, $\pi_E = \omega_{F,\xi}$, where ξ is the corresponding character of $F^\times$ via the Artin isomorphism.

From Table 4.1, $v(N) = a(\sigma'_E) = 2a(\xi) = 2$, i.e., $a(\xi) = 1$. So, the assertion (ii) is true. The property (iii) follows from Remark 4.3.11, i.e., $\xi|_{\mathfrak{o}_F^\times}$ has order $|\Lambda| = |\mathbb{Z}/4\mathbb{Z}| = 4$. For (iv), note that the representation π_E has trivial central character. By Remark 2.2.9, we have $1 = \xi|_{K^\times}(\varpi_K).\chi_{F/K}(\varpi_K) = \xi(\varpi_K)\cdot(-1) \quad (\text{since } \chi_{F/K}(\varpi_K) = -1)$. Since F is the unramified extension over K, we may assume $\varpi_K = \varpi_F$. Hence, $\xi(\varpi_F) = -1$. The statement (v) is immediate from Remark 2.2.9.

When E satisfies S_4, we have a uniqueness result. Since $a(\xi) = 1$, we get the induced character $\xi : \mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \to \mathbb{C}^\times$ of order 4. Note that $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$ is a cyclic group of order $3^{2n}-1$, where K has degree n over $\mathbb{Q}_3$, and $4 \mid 3^{2n}-1$. Now, there are exactly 2 elements of order 4 in $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$ which are inverses of each other. Since $\xi^\sigma = \xi^{-1}$ and $\xi^\sigma \neq \xi$, there are exactly two such characters in this case, which are Galois conjugates of each other. The representation π_E is a **triply imprimitive representation**, since $\xi^2 = (\xi^\sigma)^2$ and $a(\pi_E) = 2$ (see Corollary 3.2 from [45]). In fact by Theorem 4.1 from [45], we see that π_E is a unique such representation and $q \equiv 3 \bmod 4$ (i.e., n is odd).

Now assume that E satisfies the condition S_m for $m = 3$ or 6. In these two cases, $\Delta \notin K^{\times 2}$ and $v(\Delta)$ is even. So, $\Delta = \varpi^{2k}\cdot u$ for some $u \in \mathfrak{o}_K^\times \setminus {\mathfrak{o}_K}^{\times 2}$. Hence, the field $F = K(\sqrt{\Delta}) = K(\sqrt{u})$ is the unique unramified extension of K of degree 2. Since $\Delta \notin K^{\times 2}$, by Proposition 3.2 of [24], $W(L/K) = \mathbb{Z}/m\mathbb{Z} \rtimes \langle\Phi\rangle$ is not abelian. Now, since Φ^2 acts trivially on $\mathbb{Z}/m\mathbb{Z}$, we have $W(L/F) = \mathbb{Z}/m\mathbb{Z} \times \langle\Phi^2\rangle$. Then, using a similar argument as in the case of S_4, $\pi_E = \omega_{F,\xi}$ where ξ is the corresponding character of $F^\times$. The other assertions also follow from some similar arguments used in the case of S_4.

Case 2: Suppose that E satisfies either the condition S'_6 or S''_6. Then by Théorème 1 of [26] (also see Theorem 3.1 of [24]), we have $\Lambda \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. We will give a proof for the condition $\mathrm{S}_6^{'}$; the other case is similar. For this case we consider the diagram on the right of (4.15). If E satisfies the condition $\mathrm{S}_6^{'}$, then $\Delta \notin K^{\times 2}$ and $v(\Delta)$ is odd. So, $\Delta = \varpi^{2l+1} \cdot u$ for some $u \in \mathfrak{o}_K^{\times}$, and

$$K(\sqrt{\Delta}) = \begin{cases} K(\sqrt{\varpi}) & \text{if } u \in \mathfrak{o}_K^{\times 2}, \\ K(\sqrt{\varpi u}) & \text{if } u \notin \mathfrak{o}_K^{\times 2}. \end{cases}$$

In either case, $K(\sqrt{\Delta})$ is a ramified extension of K of degree 2. By Proposition 3.2 of [24], $W(L/K(\sqrt{\Delta}))$ is abelian since $\Delta \in K(\sqrt{\Delta})^{\times 2}$. Let $F = K(\sqrt{\Delta})$. Since F^{un} is the compositum of F and K^{un}, we get $\mathrm{Gal}(K^{\mathrm{un}}/K) \cong \mathrm{Gal}(F^{\mathrm{un}}/F) = \langle \Phi \rangle$. Here we consider an inverse Frobenius $\Phi \in \mathrm{Gal}(F^{\mathrm{un}}/F)$ as an image of an inverse Frobenius of $\mathrm{Gal}(\bar{K}/K)$ inside $\mathrm{Gal}(F^{\mathrm{un}}/F)$. Now, $\mathrm{Gal}(L/F^{\mathrm{un}}) \cong \mathbb{Z}/6\mathbb{Z}$ is the unique subgroup of order 6 in Λ. Hence, we get $W(L/F) = \mathbb{Z}/6\mathbb{Z} \times \langle \Phi \rangle$. Then, using a similar argument as in case 1 we get $\pi_E = \omega_{F,\xi}$, where the field F is a ramified quadratic extension of K and ξ is the corresponding character of $F^{\times}$. For property (ii), we use the relation $v(N) = a(\sigma_E') = 1 + a(\xi)$ and Table 4.1. Also, one can prove the properties (iii) and (v) using similar arguments as in case 1. $\square$

The following result is a special but more precise version of Theorem 4.3.9 for $K = \mathbb{Q}_3$.

Corollary 4.3.12. *Let E be an elliptic curve over $\mathbb{Q}_3$ given by a minimal Weierstrass equation of the form (2.71) with the coefficients in $\mathbb{Z}_3$. Assume that E satisfies one of the conditions in $\{\mathrm{S}_4, \mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6^{'}, \mathrm{S}_6^{''}\}$ as defined in Table 4.1. Then the corresponding $\mathrm{GL}(2, \mathbb{Q}_3)$ representation π_E is a supercuspidal representation, i.e., $\pi_E = \omega_{F,\xi}$, where F is a quadratic extension of $\mathbb{Q}_3$ and ξ is a character of $F^{\times}$ with the following properties:*

(i)

$$F = \begin{cases} \mathbb{Q}_3(i) & \textit{is the unramified extension of } \mathbb{Q}_3 \textit{ if } E \textit{ satisfies } \mathrm{S}_4, \\ \mathbb{Q}_3(\sqrt{\Delta}) & \textit{is the unramified extension of } \mathbb{Q}_3 \textit{ if } E \textit{ satisfies } \mathrm{S}_3 \textit{ or } \mathrm{S}_6, \\ \mathbb{Q}_3(\sqrt{\Delta}) & \textit{is a ramified extension of } \mathbb{Q}_3 \textit{ if } E \textit{ satisfies } \mathrm{S}_6^{'} \textit{ or } \mathrm{S}_6^{''}. \end{cases}$$

(ii) The conductor $a(\xi)$ of ξ is given by

$$a(\xi) = \begin{cases} 1 & \textit{if } E \textit{ satisfies } \mathrm{S}_4, \\ 2 & \textit{if } E \textit{ satisfies a condition in } \left\{\mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6^{'}\right\}, \\ 4 & \textit{if } E \textit{ satisfies } \mathrm{S}_6^{''}. \end{cases}$$

(iii) If E satisfies the condition S_m for $m \in \{3,4,6\}$, then $\xi|_{\mathfrak{o}_F^\times}$ has order m. If E satisfies the condition $\mathrm{S}_6^{'}$ or $\mathrm{S}_6^{''}$, then $\xi|_{\mathfrak{o}_F^\times}$ has order 6.

(iv) For all cases we have $\xi^\sigma = \xi^{-1}$ on $F^\times$.

(v) Furthermore, when E satisfies one of the conditions in $\{\mathrm{S}_4, \mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6^{'}\}$, there are exactly two such characters on $\mathfrak{o}_F^\times$ which are Galois conjugates of each other. When E satisfies $\mathrm{S}_6^{''}$, there are exactly six such characters on $\mathfrak{o}_F^\times$.

Note that most of the assertions of Corollary 4.3.12 follow from the proof of Theorem 4.3.9 except for the property (v). We give a proof for the property (v) in the next subsection using a few lemmas on the structures of some groups over a p-adic field.

4.3.3 Structures of some groups over a p-adic field

Lemma 4.3.13. *Let F be any non-archimedean local field of characteristic 0. Then $\mathfrak{o}_F/\mathfrak{p}_F^m \cong (1+\mathfrak{p}_F^n)/(1+\mathfrak{p}_F^{n+m})$ for $n \geq m \geq 1$.*

Proof. Consider the map $\mathfrak{o}_F \xrightarrow{f} (1+\mathfrak{p}_F^n)/(1+\mathfrak{p}_F^{n+m})$ such that $f(x) = 1 + \varpi_F^n x$.

$$\begin{aligned}
f(x+y) &= 1 + \varpi_F^n(x+y) \\
&= (1+\varpi_F^n x)(1+\varpi_F^n y) - \varpi_F^{2n}xy \\
&= (1+\varpi_F^n x)(1+\varpi_F^n y)\left(1 - \frac{\varpi_F^{2n}xy}{(1+\varpi_F^n x)(1+\varpi_F^n y)}\right) \\
&= (1+\varpi_F^n x)(1+\varpi_F^n y)\ \left(\frac{\varpi_F^{2n}xy}{(1+\varpi_F^n x)(1+\varpi_F^n y)} \in \mathfrak{p}_F^{2n} \subset \mathfrak{p}_F^{n+m}\right) \\
&= f(x)f(y).
\end{aligned}$$

So, f is a homomorphism. Clearly, f is surjective and its kernel is $\mathfrak{p}_F^m$. Hence, f defines an isomorphism between $\mathfrak{o}_F/\mathfrak{p}_F^m$ and $(1+\mathfrak{p}_F^n)/(1+\mathfrak{p}_F^{n+m})$ for $n \geq m \geq 1$. $\square$

Lemma 4.3.14. *Let F be a quadratic extension of $\mathbb{Q}_3$. Then the following exact sequence*

$$1 \to (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^2) \cong \mathfrak{o}_F/\mathfrak{p}_F \xrightarrow{\alpha} \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \xrightarrow{\beta} \mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \to 1$$

splits, i.e., $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \cong (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^2) \times \mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$. Here α is defined by $\alpha(x) = x$ and β is defined by $\beta\left(x(1+\mathfrak{p}_F^2)\right) = x(1+\mathfrak{p}_F)$.

Proof. Note that, $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$ is a cyclic group of order 3^2-1 (resp. 2) if $F/\mathbb{Q}_3$ is unramified (resp. ramified). Let $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F) = \langle g(1+\mathfrak{p}_F)\rangle$. Evidently, β is surjective and $\beta\left(g(1+\mathfrak{p}_F^2)\right) = g(1+\mathfrak{p}_F)$. Now we have

$$\begin{aligned}
&o\left(g(1+\mathfrak{p}_F)\right) \mid o\left(g(1+\mathfrak{p}_F^2)\right) \quad \text{(here } o(g) \text{ denotes the order of } g.\text{)} \\
\Rightarrow\ &o\left(g(1+\mathfrak{p}_F^2)\right) = m\cdot(3^i-1),\ \text{where } m \mid 3^i \text{ and } \gcd(m, 3^i-1) = 1,\ i = 1, 2.
\end{aligned}$$

Then $o\left(g^m(1+\mathfrak{p}_F^2)\right) = (3^i-1)$ for $i = 1$ or 2. Let $y = g^m \in \mathfrak{o}_F^\times$.
We define, $\beta' : \mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \to \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2)$ by $\beta'\left(y(1+\mathfrak{p}_F)\right) = y(1+\mathfrak{p}_F^2)$. Then $(\beta\circ\beta')\left(y(1+\mathfrak{p}_F)\right) = y(1+\mathfrak{p}_F)$, i.e., $\beta\circ\beta' = \mathrm{id}_{\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)}$. Hence, the given exact sequence splits. $\square$

Lemma 4.3.15. *Let F be a ramified quadratic extension of $\mathbb{Q}_3$. Then the following exact sequence*

$$1 \to (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4) \xrightarrow{\alpha_1} \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^4) \xrightarrow{\beta_1} \mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \to 1 \tag{4.18}$$

splits, i.e., $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^4) \cong (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4)\times \mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$. Here α_1 is defined by $\alpha_1(x) = x$ and β_1 is defined by $\beta_1\left(x(1+\mathfrak{p}_F^4)\right) = x(1+\mathfrak{p}_F)$.

Proof. Since $\gcd(2, |1+\mathfrak{p}_F/1+\mathfrak{p}_F^2|) = 1$ and $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$ is a cyclic group of order 2, one can give arguments similar as in the proof of Lemma 4.3.14 to show that (4.18) splits. $\square$

Lemma 4.3.16. *Let $F = \mathbb{Q}_3(\sqrt{-3})$. Then $(1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4) \cong (\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. In order to prove the statement, we count the number of elements of order 3 in $(1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4)$. Since F is a ramified quadratic extension of $\mathbb{Q}_3$, $v_F(3) = 2$. Let $3 = \varpi_F^2 u$ for some $u \in \mathfrak{o}_F^\times$. Since $F = \mathbb{Q}_3(\sqrt{-3})$, without loss of generality, we can choose $\varpi_F = \sqrt{-3}$. This implies $u = -1$. Let $x = 1 + \varpi_F y \in (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4)$ where $y \in \mathfrak{o}_F$. Then

$$\begin{aligned}
x^3 = 1 \text{ in } (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4) &\Leftrightarrow x^3 \in 1+\mathfrak{p}_F^4. \\
&\Leftrightarrow (1+\varpi_F y)^3 \in 1+\mathfrak{p}_F^4. \\
&\Leftrightarrow 1 + \varpi_F^3 y^3 + 3\varpi_F y + 3\varpi_F^2 y^2 \in 1+\mathfrak{p}_F^4. \\
&\Leftrightarrow \varpi_F^3 y^3 + \varpi_F^3 u y \in \mathfrak{p}_F^4. \\
&\Leftrightarrow y^3 - y \in \mathfrak{p}_F. \\
&\Leftrightarrow y(y-1)(y+1) \in \mathfrak{p}_F.
\end{aligned}$$

The last statement is always true. Hence, $(1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4) \cong (\mathbb{Z}/3\mathbb{Z})^3$. $\square$

Lemma 4.3.17. *Suppose $E/\mathbb{Q}_3$ satisfies* S_6'' *from Table 4.1. Then the representation π_E of* $\mathrm{GL}(2,\mathbb{Q}_3)$ *associated to $E/\mathbb{Q}_3$ is $\pi_E = \omega_{F,\xi}$ with $F = \mathbb{Q}_3(\sqrt{-3})$.*

Proof. When E satisfies S_6'', by (i) of Corollary 4.3.12, we have $\pi_E = \omega_{F,\xi}$ with $F = \mathbb{Q}_3(\sqrt{\Delta})$. To see that $F = \mathbb{Q}_3(\sqrt{-3})$, we need to consider the following possible values of $(v_3(\Delta), v_3(c_4), v_3(c_6))$ from Table 4.1 when E satisfies S_6''

$$(v_3(\Delta), v_3(c_4), v_3(c_6)) \in \{(5, \geq 3, 4), (7, \geq 4, 5), (11, \geq 5, 7), (13, \geq 6, 8)\} \tag{4.19}$$

and use the relation $\Delta = \frac{c_4^3 - c_6^2}{1728}$. Here we check one of the cases. Assume that $v_3(\Delta) = 5$, $v_3(c_4) \geq 3$ and $v_3(c_6) = 4$. We have $c_4 = 3^m c_4'$ for $m \geq 3$ and $c_6 = 3^4 c_6'$, where $c_4', c_6' \in \mathbb{Z}_3^\times$. Then

$$\Delta = \frac{c_4^3 - c_6^2}{1728} \Rightarrow \quad \Delta = \frac{(3^m c_4')^3 - (3^4 c_6')^2}{3^3 \cdot 2^6} \Rightarrow \quad \Delta = \frac{3^{3m} c_4'^3 - 3^8 c_6'^2}{3^3 \cdot 2^6}.$$

$$\Rightarrow \quad 2^6 \Delta = 3^5 \left(3^{3m-8} c_4'^3 - c_6'^2\right) \Rightarrow \quad 2^6 \frac{\Delta}{3^5} = 3^{3m-8} c_4'^3 - c_6'^2. \quad (3^{3m-8} c_4'^3 - c_6'^2 \in \mathbb{Z}_3^\times.)$$

Since $3^{3m-8} c_4'^3 \equiv 0 \bmod 3$ and $c_6'^2 \equiv 1 \bmod 3$, we get

$$\frac{\Delta}{3^5} \equiv -1 \bmod 3, \quad \text{i.e.,} \quad \frac{\Delta}{3^{v_3(\Delta)}} \equiv -1 \bmod 3.$$

Similarly, one can check that $3^{-v_3(\Delta)} \Delta \equiv -1 \bmod 3$ for all other cases too. Hence, $F = \mathbb{Q}_3(\sqrt{-3})$. $\square$

Proof of the property (v) in Corollary 4.3.12. We consider three cases.

Case 1: Assume that E satisfies S_3 or S_6. Then by the property (i), $F = \mathbb{Q}_3(\sqrt{\Delta})$ is the unramified quadratic extension of $\mathbb{Q}_3$. Also, $a(\xi) = 2$ and the order of $\xi|_{\mathfrak{o}_F^\times}$ is 3 or 6.

By Lemma 4.3.13, $(1 + \mathfrak{p}_F)/(1 + \mathfrak{p}_F^2) \cong \mathfrak{o}_F/\mathfrak{p}_F$ has order 3^2. Since the characteristic of F is 3, we get $3x = 0$, for all $x \in \mathfrak{o}_F/\mathfrak{p}_F$. Then $(1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^2) \cong \mathfrak{o}_F/\mathfrak{p}_F \cong (\mathbb{Z}/3\mathbb{Z})^2$.

Also, $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F)$ is the cyclic group of order $3^2-1=8$. Using Lemma 4.3.14, we get

$$\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \cong (\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/8\mathbb{Z}. \tag{4.20}$$

Since $a(\xi)=2$, we get the induced character $\xi : \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \cong (\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{C}^\times$. Again, using Lemma 4.3.14, $\mathfrak{o}_K^\times/(1+\mathfrak{p}^2) \cong \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Now, $\mathfrak{o}_K^\times/(1+\mathfrak{p}^2) \hookrightarrow \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2)$ (because $\mathfrak{o}_K^\times \cap (1+\mathfrak{p}_F^2) = (1+\mathfrak{p}^2)$). Since $\chi_{F/K}$ is the unramified quadratic character of $K^\times$, using Remark 2.2.9, $\xi|_{\mathfrak{o}_K^\times/(1+\mathfrak{p}^2)} = 1$. Then, we get the induced character

$$\xi : \left(\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2)\right) / \left(\mathfrak{o}_K^\times/(1+\mathfrak{p}^2)\right) \cong \mathbb{Z}/12\mathbb{Z} \longrightarrow \mathbb{C}^\times \tag{4.21}$$

of order 3 or 6 when E satisfies S_3 or S_6 respectively. Now, $\mathbb{Z}/12\mathbb{Z}$ has exactly 2 elements of order 3 (resp. order 6), which are inverses of each other. So, using property (iv) in Corollary 4.3.12, we conclude that there are exactly two such characters of order 3 (resp. order 6) on $\mathfrak{o}_F^\times$ when E satisfies S_3 (resp. S_6), and they are Galois conjugates of each other.

Case 2: Assume that E satisfies S_6'. Then $F=\mathbb{Q}_3(\sqrt{\Delta})$ is a ramified quadratic extension of $\mathbb{Q}_3$ and ξ is a character of $F^\times$ such that the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6, and $a(\xi)=2$. Now, we have

$$\mathfrak{o}_F/\mathfrak{p}_F \cong \mathbb{Z}_3/3\mathbb{Z}_3 \cong \mathbb{Z}/3\mathbb{Z} \text{ and } \mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \cong (\mathbb{Z}_3/3\mathbb{Z}_3)^\times \cong \mathbb{Z}/2\mathbb{Z}. \tag{4.22}$$

So, by Lemma 4.3.14, $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \cong \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$. Since $a(\xi)=2$, we get the induced character

$$\xi : \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^2) \cong \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{C}^\times \tag{4.23}$$

of order 6. There are exactly $\varphi(6)=2$ elements of order 6 in $\mathbb{Z}/6\mathbb{Z}$, which are inverses of each other. Hence, using property (iv) in Corollary 4.3.12, there are exactly two such characters of order 6 on $\mathfrak{o}_F^\times$ when E satisfies S_6', and they are Galois conjugates of each other.

Case 3: Assume that E satisfies S_6''. Then, by Lemma 4.3.17, $F = \mathbb{Q}_3(\sqrt{-3})$. In this case, ξ is a character of $F^\times$ such that the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6 and $a(\xi) = 4$. So, we get the induced character $\xi : \mathfrak{o}_F^\times/(1+\mathfrak{p}_F^4) \longrightarrow \mathbb{C}^\times$. Since $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \cong \mathbb{Z}/2\mathbb{Z}$, using Lemma 4.3.15 and Lemma 4.3.16, we get $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^4) \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3$. Now, $(1+\mathfrak{p})/(1+\mathfrak{p}^2) \hookrightarrow (1+\mathfrak{p}_F)/(1+\mathfrak{p}_F^4)$ (since $\mathfrak{p} \cap \mathfrak{p}_F^4 = \mathfrak{p}^2$). By corollary 3 of §V3 in [46], $1+\mathfrak{p} = N_{F/K}(1+\mathfrak{p}_F^2)$. Since $\xi^\sigma = \xi^{-1}$ on $F^\times$, we get $\xi(1+\mathfrak{p}) = \xi\left(N_{F/K}(1+\mathfrak{p}_F^2)\right) = 1$. So, $\xi|_{(1+\mathfrak{p})/(1+\mathfrak{p}^2)} = 1$, where $(1+\mathfrak{p})/(1+\mathfrak{p}^2) \cong \mathbb{Z}/3\mathbb{Z}$. Then we get the following induced character of order 6

$$\xi : \left(\mathfrak{o}_F^\times/(1+\mathfrak{p}_F^4)\right) / \left((1+\mathfrak{p})/(1+\mathfrak{p}^2)\right) \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2 \longrightarrow \mathbb{C}^\times.$$

Also, ξ is nontrivial on $(1+\mathfrak{p}_F^3)/(1+\mathfrak{p}_F^4)$, since $a(\xi) = 4$. So, there are exactly 6 characters ξ such that $a(\xi) = 4$, the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6, and $\xi^\sigma = \xi^{-1}$ on $F^\times$. This completes the proof of the property (v) of Corollary 4.3.12. $\square$

4.4 Potential good reduction in residual characteristic 3 and $v(3) > 1$

Let K be a non-archimedean local field of characteristic 0 and residual characteristic 3. Let $\mathfrak{o}_K$ be the ring of integers of K and $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Let ϖ_K be a generator of $\mathfrak{p}$ and $k = \mathfrak{o}_K/\mathfrak{p}$ be the residue field of K of order q. Let $v : K \to \mathbb{Z}$ be the normalized valuation on K. Let E be an elliptic curve over K with additive but potential good reduction. Let us assume that $v(3) = \lambda > 1$ in K, i.e., K is a ramified extension of $\mathbb{Q}_3$.

Suppose that E/K is given by a Weierstrass equation of the form (2.71). The discriminant Δ, the j-invariant $j(E)$, c_4 and c_6 are defined in terms of the Weierstrass coefficients as in (2.72). In this section, we describe the representation π_E of $\mathrm{GL}(2,K)$ associated with E/K. In Table 4.2, we list all the possible Néron types of E/K in terms of the quantities $v(\Delta), v(c_4)$ and $v(c_6)$. Similar as Lemma 4.3.1, in this case also we can show that E has additive but potential good reduction if and only if E satisfies one and only one of the conditions in Table 4.2. This table is reproduced from Table III of [33]. For $i \in \{2,3,5,6\}$, the condition R_i in Table 4.2 is defined as follows

$$\mathrm{R}_i: \quad x^3 - 3c_4x - 2c_6 \equiv 0 \bmod (27\varpi_{\mathrm{K}}^{\mathrm{i}}) \text{ for some x} \in \mathfrak{o}_{\mathrm{K}}. \tag{4.24}$$

If an elliptic curve E/K does not satisfy the condition R_i, then we denote it by "non R_i" in Table 4.2. We now give a description of π_E in terms of the Néron type of E/K in Theorem 4.4.1, but, it is hard to describe π_E in a simple way in terms of the Weierstrass coefficients of E/K. The description is a bit complicated and not useful for our study, so we skip the proof (one can give arguments similar as in the proofs of Theorem 4.3.4 and Theorem 4.3.9).

Theorem 4.4.1. *Let K be a non-archimedean local field of characteristic 0 and residual characteristic 3 such that $v(3) > 1$ in K. Let E be an elliptic curve over K given by a minimal Weierstrass equation of the form (2.71). Let π_E be the representation of $\mathrm{GL}(2,K)$ attached to E/K. Suppose that E has additive but potential good reduction. Then π_E is one of the following*

- *If E has the Néron type I_0^*, then $\pi_E = \chi \times \chi^{-1}$ such that $a(\chi) = 1$ and $\chi|^2_{\mathfrak{o}_K^\times} = 1$.*
- *If $-1 \in k^{\times 2}$ and E has the Néron type III or III*, then $\pi_E = \chi \times \chi^{-1}$ such that $a(\chi) = 1$ and $\chi|^4_{\mathfrak{o}_K^\times} = 1$.*
- *If $-1 \notin k^{\times 2}$ and E has the Néron type III or III*, then $\pi_E = \omega_{F,\xi}$, where $F = K(i)$ is the unramified quadratic extension of K such that $a(\xi) = 1$ and $\xi|^4_{\mathfrak{o}_F^\times} = 1$.*

Table 4.2: Néron types in terms of the quantities $v(\Delta), v(c_4), v(c_6)$ when $v(3) > 1$.

$$\text{Here } \lambda = v(3), l = v(b_2), m = v(b_4), \text{ and } E(a,b) = \begin{cases} \inf\,(a,b) & \text{if } a \neq b, \\ \geq a & \text{if } a = b. \end{cases}$$

$v(\Delta)$	$v(c_4)$	$v(c_6)$	Condition on c_4, c_6	Values of l, m	Néron type	$v(N)$
6	$E(2l, \lambda+2)$	$3l$	R_3	$1 \leq l \leq \lambda$	I_0^*	2
	$\lambda+2$	$\geq 3\lambda+3$				
3	$E(2l, \lambda+1)$	$3l$	R_2	$1 \leq l \leq \lambda$	III	2
	$\lambda+1$	$\geq 3\lambda+2$				
9	$E(2l, \lambda+3)$	$3l$	R_5	$2 \leq l \leq \lambda+1$	III^*	
	$\lambda+3$	$\geq 3\lambda+5$				
$3m$	$E(2l, \lambda+m)$	$3l$	non R_2	$1 \leq m \leq l \leq \lambda$	II	$3m$
$3m$	$\lambda+m$	$3\lambda+1$		$1 \leq m \leq \lambda$		$3m$
$3l+1$	$2l$	$3l$		$1 \leq l \leq \lambda$		$3l+1$
$3\lambda+2$	$\geq 2\lambda+1$	$3\lambda+1$				$3\lambda+2$
$3m$	$E(2l, \lambda+m)$	$3l$	R_5 non R_6	$4 \leq m \leq l+1 \leq \lambda+2$	III^*	$3m-8$
$3m$	$\lambda+m$	$3\lambda+5$		$4 \leq m \leq \lambda+3$		$3m-8$
$3l+5$	$2l$	$3l$		$2 \leq l \leq \lambda+1$		$3l-3$
$3\lambda+10$	$\geq 2\lambda+4$	$3\lambda+5$				$3\lambda+2$
$3m$	$E(2l, \lambda+m)$	$3l$	R_2 non R_3	$2 \leq m \leq l \leq \lambda$	IV	$3m-2$
$3m$	$\lambda+m$	$3\lambda+2$		$2 \leq m \leq \lambda+1$		$3m-2$
$3l+2$	$2l$	$3l$		$1 \leq l \leq \lambda$		$3l$
$3\lambda+4$	$\geq 2\lambda+2$	$3\lambda+2$				$3\lambda+2$
$3m$	$E(2l, \lambda+m)$	$3l$	R_3 non R_5	$3 \leq m \leq l+1 \leq \lambda+2$	IV^*	$3m-6$
$3m$	$\lambda+m$	$3\lambda+4$		$3 \leq m \leq \lambda+2$		$3m-6$
$3l+4$	$2l$	$3l$		$2 \leq l \leq \lambda+1$		$3l-2$
$3\lambda+8$	$\geq 2\lambda+3$	$3\lambda+4$				$3\lambda+2$

- *If* $\Delta \in K^{\times 2}$, $v(\Delta) \equiv 0 \bmod 4$ *and* E *has one of the Néron types* II, II*, IV *or* IV*, *then* $\pi_E = \chi \times \chi^{-1}$ *such that* $\chi|^3_{\mathfrak{o}_K^\times} = 1$ *and* $a(\chi)$ *is given by* (4.25).

$$a(\chi) = \begin{cases} \frac{v(\Delta)}{2} & \text{if the Néron type of } E \text{ is II,} \\ \frac{v(\Delta)-8}{2} & \text{if the Néron type of } E \text{ is II}^*, \\ \frac{v(\Delta)-2}{2} & \text{if the Néron type of } E \text{ is IV,} \\ \frac{v(\Delta)-6}{2} & \text{if the Néron type of } E \text{ is IV}^*. \end{cases} \tag{4.25}$$

- *If* $\Delta \in K^{\times 2}$, $v(\Delta) \equiv 2 \bmod 4$ *and* E *has one of the Néron types* II, II*, IV *or* IV*, *then* $\pi_E = \chi \times \chi^{-1}$ *such that* $\chi|^6_{\mathfrak{o}_K^\times}$ *and* $a(\chi)$ *is given by* (4.25).

- *If* $\Delta \notin K^{\times 2}$, $v(\Delta) \equiv 0 \bmod 4$ *and* E *has one of the Néron types* II, II*, IV *or* IV*, *then* $\pi_E = \omega_{F,\xi}$, *where* $F = K(\sqrt{\Delta})$ *is the unramified quadratic extension of* K *with* $\xi|^3_{\mathfrak{o}_F^\times} = 1$ *and* $a(\xi)$ *is given by the formula* (4.25) *with* $a(\chi)$ *replaced by* $a(\xi)$.

- *If* $\Delta \notin K^{\times 2}$, $v(\Delta) \equiv 2 \bmod 4$ *and* E *has one of the Néron types* II, II*, IV *or* IV*, *then* $\pi_E = \omega_{F,\xi}$, *where* $F = K(\sqrt{\Delta})$ *is the unramified quadratic extension of* K *with* $\xi|^6_{\mathfrak{o}_F^\times} = 1$ *and* $a(\xi)$ *is given by the formula* (4.25) *with* $a(\chi)$ *replaced by* $a(\xi)$.

- *If* $\Delta \notin K^{\times 2}$, $v(\Delta) \equiv 1 \bmod 2$ *and* E *has one of the Néron types* II, II*, IV *or* IV*, *then* $\pi_E = \omega_{F,\xi}$, *where* $F = K(\sqrt{\Delta})$ *is a ramified quadratic extension of* K *such that* $\xi|^6_{\mathfrak{o}_F^\times} = 1$ *and* $a(\xi)$ *is given by* (4.26).

$$a(\xi) = \begin{cases} v(\Delta) - 1 & \text{if the Néron type of } E \text{ is II,} \\ v(\Delta) - 9 & \text{if the Néron type of } E \text{ is II}^*, \\ v(\Delta) - 3 & \text{if the Néron type of } E \text{ is IV,} \\ v(\Delta) - 7 & \text{if the Néron type of } E \text{ is IV}^*. \end{cases} \tag{4.26}$$

Chapter 5

Symmetric cube of local representations attached to elliptic curves

As before, let K be a non-achimedean local field of characteristic 0 and residual characteristic p. Let $\mathfrak{o}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}_K$. Suppose that $v : K \to \mathbb{Z}$ is the normalized valuation on K, and $k = \mathfrak{o}_K/\mathfrak{p}$ is the residue field of K of order q. We fix a generator ϖ_K for the ideal $\mathfrak{p}$ and ϖ_K is called the uniformizer. We write $\nu(x)$ or $|x|$ for the normalized absolute value of x; thus $\nu(\varpi_K) = q^{-1}$. Let π_E be the irreducible, admissible representation of $\mathrm{GL}(2, K)$ attached to an elliptic curve E over K such that it is a local component of a cuspidal automorphic representation of $\mathrm{GL}(2)$. In this chapter, we study the L-packet $\mathrm{sym}^3(\pi_E)$. We have discussed the local sym^3 lifting in general in Section 3.3. But here we specifically study the L-factor, ε-factor, and representation type of the L-packet $\mathrm{sym}^3(\pi_E)$ when π_E is attached to an elliptic curve E/K. We assume that $v(3) = 1$ if the residual characteristic of K is 3.

5.1 Type of representation of $\mathrm{sym}^3(\pi_E)$

In this section, we find the representation type of $\mathrm{sym}^3(\pi_E)$ in Table 5.1 and Table 5.2. Here we use our discussion in Section 3.3 and Table 3.1 in order to determine the representation type of $\mathrm{sym}^3(\pi_E)$. We consider three cases.

Case I: Assume that E has potential multiplicative reduction over any p-adic field K. Then, by Theorem 4.1.1, we have $\pi_E = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GL}(2)}$, where $(\gamma(E/K), \cdot)$ is the quadratic character of $K^\times$ defined by the Hilbert symbol $(\cdot,\cdot)$. Then $\mathrm{sym}^3(\pi_E) = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GSp}(4)}$. So, by Table 3.1, $\mathrm{sym}^3(\pi_E)$ is of type **IVa**. Thus we obtain the first row of Table 5.1.

Case II: Assume that E has additive but potential good reduction over a p-adic field K with $p \geq 3$ and $\pi_E = \chi \times \chi^{-1}$. Then we have $\mathrm{sym}^3(\pi_E) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$. Then, by Table 3.1, $\mathrm{sym}^3(\pi_E)$ is of type **I**. Thus we obtain the second row of Table 5.1 and the first row of Table 5.2.

Case III: Assume that E has additive but potential good reduction over a p-adic field K with $p \geq 3$ and $\pi_E = \omega_{F,\xi}$, where F is a quadratic extension of K and ξ is a character of $F^\times$ such that $\xi^\sigma = \xi^{-1}$. Then, by Table 3.1, we have the following

- $\mathrm{sym}^3(\pi_E)$ is of type **VIII** and $\mathrm{sym}^3(\pi_E) = \{\tau(S, \omega_{F,\xi}), \tau(T, \omega_{F,\xi})\}$ if and only if $\xi^4 = 1$ on $F^\times$.
- $\mathrm{sym}^3(\pi_E)$ is of type **X** and $\mathrm{sym}^3(\pi_E) = \omega_{F,\xi^4} \rtimes \varphi$ if and only if $\xi^6 = 1$ on $F^\times$.
- $\mathrm{sym}^3(\pi_E)$ is **supercuspidal** if and only if $\xi^4 \neq 1$ and $\xi^6 \neq 1$ on $F^\times$.

Remark 5.1.1. *Suppose that $\pi_E = \omega_{F,\xi}$ such that F/K is a quadratic extension and ξ is a character of $F^\times$. Then $\xi^6 = 1$ on $F^\times$ if and only if the order of ξ is 6 and $\xi^4 = 1$ on $F^\times$ if and only if the order of ξ is 4.*

Now we consider two cases. First, assume that F is the unramified extension of K. Then we have $\xi(\varpi_F) = -1$. Since $F^\times = \langle \varpi_F \rangle \times \mathfrak{o}_F^\times$ and $\xi^2(\varpi_F) = 1$, we get $\xi^6 = 1$ on $F^\times$ if $\xi|_{\mathfrak{o}_F^\times}^3 = 1$ or $\xi|_{\mathfrak{o}_F^\times}^6 = 1$, and we get $\xi^4 = 1$ on $F^\times$ if $\xi|_{\mathfrak{o}_F^\times}^4 = 1$. Thus we get the last row of Table 5.1 when $\pi_E = \omega_{F,\xi}$, and the entries in the last column of Table 5.2 when E satisfies the condition S_m with $m \in \{3, 4, 6\}$.

Table 5.1: Representation type of $\mathrm{sym}^3(\pi_E)$.

Here E has potential multiplicative reduction or E has additive but potential good reduction with $p \geq 5$. Here $\gamma = \gamma(E/K)$ is the γ-invariant of E and $(\gamma, \cdot)$ is the Hilbert symbol and $e = \frac{12}{\gcd(12, v(\Delta))}$. Also, φ is a character of $K^\times$ such that $\xi^3 = \varphi \circ N_{F/K}$. For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition on E/K	$\mathrm{GL}(2,K)$ rep. π_E	Extra condition on π_E	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Rep. type of $\mathrm{sym}^3(\pi_E)$
$j(E) \notin \mathfrak{o}_K^\times$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GL}(2)}$		$(\gamma,\cdot)\mathrm{St}_{\mathrm{GSp}(4)}$	IVa
$j(E) \in \mathfrak{o}_K^\times$ $(p-1)v(\Delta) \equiv 0 \bmod 12$	$\chi \times \chi^{-1}$ $a(\chi) = 1$ $o\left(\chi\|_{\mathfrak{o}_K^\times}\right) = e$		$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	I
$j(E) \in \mathfrak{o}_K^\times$ $(p-1)v(\Delta) \not\equiv 0 \bmod 12$	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 1$ $\xi(\varpi_F) = -1$	$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 3$	$\omega_{F,\xi^4} \rtimes \varphi$	X
		$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 4$	$\{\tau(S,\omega_{F,\xi}), \tau(T,\omega_{F,\xi})\}$	VIII
		$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 6$	$\omega_{F,\xi^4} \rtimes \varphi$	X

Next, we assume that F is a ramified extension of K. Then the residual characteristic of K is $p = 3$ and E satisfies the condition S_6' or S_6'' from Table 4.1. In this case, the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6, but we do not have the exact value of $\xi(\varpi_F)$. Here, we investigate whether $\xi^6(\varpi_F) = 1$ or not. Note that,

$$K^\times = \langle \varpi_K \rangle \times (1 + \mathfrak{p}) \times W_{q-1} \quad \text{and} \quad F^\times = \langle \varpi_F \rangle \times (1 + \mathfrak{p}_F) \times W_{q-1}.$$

Table 5.2: Representation type of $\mathrm{sym}^3(\pi_E)$ when E/K has additive but potential good reduction and $p = 3$.

Here, φ is a character of $K^\times$ such that $\xi^3 = \varphi \circ N_{F/K}$. For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition from Table 4.1	$\mathrm{GL}(2,K)$ rep. π_E	Extra condition on K	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Representation type of $\mathrm{sym}^3(\pi_E)$
P_m $m \in \{2,3,4,6\}$	$\chi \times \chi^{-1}$ $a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = m$		$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	I
S_3	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 3$		$\omega_{F,\xi^4} \rtimes \varphi$	X
S_4	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 1$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 4$		$\{\tau(S,\omega_{F,\xi}), \tau(T,\omega_{F,\xi})\}$	VIII
S_6	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$		$\omega_{F,\xi^4} \rtimes \varphi$	X
S_6'	$\omega_{F,\xi}$ F/K is ram. $a(\xi) = 2$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	$-1 \in K^{\times 2}$	$\omega_{F,\xi^4} \rtimes \varphi$	X
		$-1 \notin K^{\times 2}$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$	Supercuspidal
S_6''	$\omega_{F,\xi}$ F/K is ram. $a(\xi) = 4$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	$-1 \in K^{\times 2}$	$\omega_{F,\xi^4} \rtimes \varphi$	X
		$-1 \notin K^{\times 2}$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$	Supercuspidal

Here, W_{q-1} is the group of $(q-1)$th root of unity. Note that $\mathfrak{o}_K^\times/(1+\mathfrak{p}) \cong W_{q-1}$. Since F/K is a ramified quadratic extension, so we also have $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \cong W_{q-1}$. Without loss of generality, we may assume that $N_{F/K}(\varpi_F) = \varpi_K$. Let $\sigma(\varpi_F) = \varpi_F y w$ for some $y \in 1+\mathfrak{p}_F$ and $w \in W_{q-1}$. Then we have the following result.

Lemma 5.1.2. *Let K be a non-archimedean field of characteristic 0 and residual characteristic 3. Let F be a ramified quadratic extension of K and σ be the non-trivial element of $\mathrm{Gal}(F/K)$. Assume that $\sigma(\varpi_F) = \varpi_F y w$, where $w \in W_{q-1}$ and $y \in 1+\mathfrak{p}_F$. Then $w = -1$.*

Proof. Firstly, we show that $w \in \{\pm 1\}$ and $y\sigma(y) = 1$ as follows.

$$\text{We have } \varpi_F = \sigma(\sigma(\varpi_F)) = \sigma(\varpi_F y w) = \sigma(\varpi_F)\sigma(y)\sigma(w) = \varpi_F y w \sigma(y)\sigma(w).$$

$$\Rightarrow y\sigma(y)w\sigma(w) = 1.$$

$$\Rightarrow y\sigma(y)w^2 = 1. \quad (w = \sigma(w) \text{ since } w \in W_{q-1}.)$$

$$\Rightarrow w^2 = 1 \text{ and } y\sigma(y) = 1. \quad (\text{since } (1+\mathfrak{p}_F) \cap W_{q-1} = 1.)$$

$$\Rightarrow w \in \{\pm 1\} \text{ and } y\sigma(y) = 1.$$

Now, we prove the lemma by contradiction. Let us assume that $w = 1$. Then, $\sigma(\varpi_F) = \varpi_F y$ with $y \in 1+\mathfrak{p}_F$, i.e., $\frac{\sigma(\varpi_F)}{\varpi_F} \in 1+\mathfrak{p}_F$. Also, $\sigma(w) = w$ for all $w \in W_{q-1}$ and $\sigma(y) \equiv y \bmod \mathfrak{p}_\mathrm{F}$ for all $y \in 1+\mathfrak{p}_F$. Then we get

$$\sigma \in R = \left\{\sigma' \in \mathrm{Gal}(F/K) : \frac{\sigma'(x)}{x} \equiv 1 \bmod \mathfrak{p}_\mathrm{F} \text{ for all } \mathrm{x} \in \mathrm{F}^\times\right\}, \tag{5.1}$$

here R is the ramification group. Now, since F/K is a quadratic extension and the residue characteristic of K is 3, we have $\gcd([F:K], 3) = 1$. So, F is a tamely ramified quadratic extension of K (see Definition 7.6 in [32]). Let V be the ramification field over K, i.e., V is the intermediate field of F/K fixed by R. Since F/K is tamely ramified, by Proposition 9.14 of [32], we get $V = F$. So, R is the trivial group in this case. This is a contradiction to (5.1). So, our assumption $w = 1$ was wrong. Hence $w = -1$. $\square$

Lemma 5.1.3. *Let E be an elliptic curve over a non-archimedean field K of characteristic 0 and residual characteristic 3. Suppose that E satisfies the condition S_6' or S_6'' in Table 4.1, so that the associated $\mathrm{GL}(2,K)$ representation is $\pi_E = \omega_{F,\xi}$, where F is a ramified quadratic extension of K and ξ is a character of $F^\times$. Then $-1 \in K^{\times 2}$ if and only if $\xi^6 = 1$ on $F^\times$.*

Proof. Using Lemma 5.1.2, we see that $\sigma(\varpi_F) = -\varpi_F y$ with $y \in 1+\mathfrak{p}_F$. Now, the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6. So, $\xi^6 = 1$ on $F^\times$ if and only if $\xi^6(\varpi_F) = 1$. Now, note that

$$\begin{aligned}
&\xi^6(\varpi_F) = 1 \Leftrightarrow \xi^3(\varpi_F) = (\xi^3)^\sigma(\varpi_F) \quad (\text{since } \xi^{-1} = \xi^\sigma.)\\
&\Leftrightarrow \xi^3(\varpi_F) = \xi^3\left(\sigma(\varpi_F)\right) \quad (\text{here } \sigma \text{ is the non-trivial element in } \mathrm{Gal}(F/K).)\\
&\Leftrightarrow \xi^3(\varpi_F) = \xi^3\left(\varpi_F\right)\xi^3\left(y\right)\xi^3\left(-1\right)\\
&\Leftrightarrow \xi^3(\varpi_F) = \xi^3\left(\varpi_F\right)\xi^3\left(-1\right) \quad (\text{since } a(\xi^3) = 1.)\\
&\Leftrightarrow \xi^3\left(-1\right) = 1 \quad (\text{since } \xi(\varpi_F) \neq 0.)
\end{aligned} \tag{5.2}$$

Let $-1 \in K^{\times 2}$. Since $\xi|_{\mathfrak{o}_F^\times}^6 = 1$ and $-1 \in \mathfrak{o}_F^{\times 2}$, so $\xi^3(-1) = 1$. Then, by (5.2), $\xi^6 = 1$ on $F^\times$. Conversely, let us assume $-1 \notin K^{\times 2}$. Since $a(\xi^3) = 1$, we have $\xi^3|_{W_{q-1}} \neq 1$. Now, $\xi^3|_{W_{q-1}^2} = 1$ (since $\xi|_{\mathfrak{o}_F^\times}^6 = 1$) and $W_{q-1} = W_{q-1}^2 \sqcup (-1)W_{q-1}^2$. So, $\xi^3(-1) = -1$. Hence, $\xi^6 \neq 1$ on $F^\times$. This concludes the lemma. $\square$

Hence, using Lemma 5.1.3, when E satisfies the condition S_6' or S_6'' in Table 4.1, $\mathrm{sym}^3(\pi_E)$ is of type **X** if $-1 \in K^{\times 2}$ and $\mathrm{sym}^3(\pi_E)$ is **supercuspidal**, if $-1 \notin K^{\times 2}$.

5.2 Conductor of $\mathrm{sym}^3(\pi_E)$

In this section, we find the conductor $a(\mathrm{sym}^3(\pi_E))$ of $\mathrm{sym}^3(\pi_E)$ in Table 5.3, Table 5.4 and Table 5.5 and list them in the following result.

Theorem 5.2.1. *let K be a non-achimedean local field of characteristic 0 and residual characteristic p, for any prime p. When the residual characteristic of K is 3, we assume that $v(3) = 1$. Let E be an elliptic curve over K given by a minimal Weierstrass equation of the form (2.71) with coefficients in $\mathfrak{o}_K$. Let π_E be the irreducible, admissible representation of $\mathrm{GL}(2, K)$ attached E.*

(i) If E/K has potential multiplicative reduction, then $a(\mathrm{sym}^3(\pi_E))$ is given in Table 5.3.

(ii) If E/K has additive but potential good reduction and $p \geq 5$, then $a(\mathrm{sym}^3(\pi_E))$ is given in Table 5.4.

(iii) If E/K has additive but potential good reduction and $p = 3$, then $a(\mathrm{sym}^3(\pi_E))$ is given in Table 5.5.

Proof. **(i)** Assume that E has potential multiplicative reduction, i.e, $j(E) \notin \mathfrak{o}_K$. By Theorem 4.1.1, the representation π_E of $\mathrm{GL}(2)$ attached to E is of the form $\pi_E = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GL}(2)}$. Then $\mathrm{sym}^3(\pi_E)$ has the L-parameter as in (3.4) and $\mathrm{sym}^3(\pi_E) = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GSp}(4)}$. Then, from Table 3.2, the conductor $a(\mathrm{sym}^3(\pi_E))$ is given by

$$a(\mathrm{sym}^3(\pi_E)) = \begin{cases} 4a((\gamma(E/K), \cdot))^3 & \text{if } (\gamma(E/K), \cdot) \text{ is ramified,} \\ 3 & \text{if } (\gamma(E/K), \cdot) \text{ is unramified.} \end{cases}$$

Since the Hilbert symbol $(\gamma(E/K), \cdot)$ takes value in $\{\pm 1\}$, we have $(\gamma(E/K), \cdot) = (\gamma(E/K), \cdot)^3$. When E/K has additive but potential multiplicative reduction, it is well known that $(\gamma(E/K, \cdot)$ is ramified and $a\left((\gamma(E/K), \cdot)\right)$ is given by

$$a\left((\gamma(E/K), \cdot)\right) = \begin{cases} 1 & \text{for } p \geq 3, \\ 2 \text{ or } 3 & \text{for } p = 2. \end{cases}$$

Table 5.3: Conductor of $\mathrm{sym}^3(\pi_E)$ when E has potential multiplicative reduction. Here $\gamma = \gamma(E/K)$ is the γ-invariant of E/K and $(\cdot,\cdot)$ is the Hilbert symbol.

Condition on E/K	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{sym}^3(\pi_E)$	Condition on π_E	Prime p	$a(\pi_E)$	$a(\mathrm{sym}^3(\pi_E))$
$j(E) \notin \mathfrak{o}_K^\times$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GL}(2)}$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GSp}(4)}$	$(\gamma,\cdot)$ is ram.	≥ 3	2	4
				2	4	8
					6	12
			$(\gamma,\cdot)$ is unr.		1	3

Table 5.4: Conductor of $\mathrm{sym}^3(\pi_E)$ when E has additive but potential good reduction and $p \geq 5$.

For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition on E/K	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{sym}^3(\pi_E)$	Condition on π_E	$a(\pi_E)$	$a(\mathrm{sym}^3(\pi_E))$
$j(E) \in \mathfrak{o}_K^\times$ $(p-1)v(\Delta)\equiv 0 \mod 12$	$\chi \times \chi^{-1}$ $a(\chi)=1$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 2$	2	4
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 3$	2	2
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 4$	2	4
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 6$	2	4
$j(E) \in \mathfrak{o}_K^\times$ $(p-1)v(\Delta)\not\equiv 0 \mod 12$	$\omega_{F,\xi}$ F/K is unr. $a(\xi)=1$ $\xi(\varpi_F)=-1$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 3$	2	2
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 4$	2	4
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	2	4

Table 5.5: Conductor of $\mathrm{sym}^3(\pi_E)$ when E has additive but potential good reduction and $p = 3$.

For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition from Table 4.1	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Condition on π_E	$a(\pi_E)$	$a(\mathrm{sym}^3(\pi_E))$
P_2	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 2$	2	4
P_3	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 2$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 3$	4	4
P_4	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 4$	2	4
P_6	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 2$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 6$	4	6
S_4	$\omega_{F,\xi}$ $F=K(i)$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)	$a(\xi) = 1$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 4$	2	4
S_3	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 3$	4	4
S_6	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	4	6
S_6'	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (ramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$a(\xi) = 2$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	3	5
S_6''	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (ramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$a(\xi) = 4$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	5	7

Hence, we obtain the column of $a(\mathrm{sym}^3(\pi))$ of Table 5.3, i.e.,

$$a(\mathrm{sym}^3(\pi_E)) = \begin{cases} 4 & \text{if } (\gamma(E/K), \cdot) \text{ is ramified and } p \geq 3, \\ 8 \text{ or } 12 & \text{if } (\gamma(E/K), \cdot) \text{ is ramified and } p = 2, \\ 3 & \text{if } (\gamma(E/K), \cdot) \text{ is unramified.} \end{cases}$$

(ii) Let $e = \frac{12}{\gcd(12, v(\Delta))}$. Assume that E has additive but potential good reduction, i.e, $j(E) \in \mathfrak{o}_K$. Then we consider the following two cases.

Case 1: Suppose that $(q-1)v(\Delta) \equiv 0 \bmod 12$. By Theorem 4.2.1, the corresponding $\mathrm{GL}(2, K)$ representation is of the form $\pi_E = \chi \times \chi^{-1}$, where χ is a character of $K^\times$. Then, using Table 3.1, we get $\mathrm{sym}^3(\pi_E) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$. Then, from Table 3.2, the conductor $a(\mathrm{sym}^3(\pi_E))$ of $\mathrm{sym}^3(\pi_E)$ is given by $a(\mathrm{sym}^3(\pi_E)) = 2a(\chi^3) + 2a(\chi)$. Now, by Theorem 4.2.1, $a(\chi) = 1$ and χ^e is unramified for $e \in \{2, 3, 4, 6\}$. When $e = 2$ or 4, we have $a(\chi^3) = a(\chi)$, so we get $a(\mathrm{sym}^3(\pi_E)) = 4a(\chi) = 4$. When $e = 3$, $a(\mathrm{sym}^3(\pi_E)) = 2a(\chi) = 2$. When $e = 6$, note that $\chi^3|_{\mathfrak{o}_K^\times} \neq 1$, so we get $0 < a(\chi^3) \leq a(\chi) = 1$. Since the residual characteristic p of K is odd, we have $1 + \mathfrak{p} \subset \mathfrak{o}_K^{\times 2}$. Now, $\chi^6|_{\mathfrak{o}_K^\times} = 1$ implies $\chi^3|_{1+\mathfrak{p}} = 1$, i.e., $a(\chi^3) = 1$. So, $a(\mathrm{sym}^3(\pi_E)) = 4$ when $e = 6$. Hence, we obtain the first row of Table 5.4.

Case 2: Suppose that $(q-1)v(\Delta) \not\equiv 0 \bmod 12$. By Theorem 4.2.2, the corresponding $\mathrm{GL}(2, K)$ representation is of the form $\pi_E = \omega_{F,\xi}$, where F is the unramified quadratic extension of K and ξ is a character of $F^\times$ with $\xi^\sigma = \xi^{-1}$. Also, the central character of π_E is trivial. Then, using Table 3.1, the local parameter of $\mathrm{sym}^3(\pi_E)$ is of the form $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi) \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ and $\mathrm{sym}^3(\pi_E) = \omega_{F,\xi} \oplus \omega_{F,\xi^3}$. So, using Table 3.2, we get $a(\mathrm{sym}^3(\pi_E)) = 2a(\xi^3) + 2a(\xi)$.

By Theorem 4.2.2, in this case $a(\xi) = 1$ and ξ^e is unramified for $e \in \{3, 4, 6\}$. Clearly, if $e = 3$, we have $a(\xi^3) = 0$ and $a(\mathrm{sym}^3(\pi_E)) = 2$ and if $e = 4$, $a(\mathrm{sym}^3(\pi_E)) = 4a(\xi) = 4$. Now, since the residual characteristic of F is also odd, we have $1 + \mathfrak{p}_F \subset \mathfrak{o}_F^{\times 2}$. Since

$0 \leq a(\xi^3) \leq a(\xi)$ and $\xi|^3_{\mathfrak{o}_F^\times} \neq 1$, using a similar argument as in case 1, it is easy to check that $a(\xi^3) = 1$ and $a(\mathrm{sym}^3(\pi_E)) = 4$ if $e = 6$. This proves the second row of Table 5.4.

(iii) The proof of this part is similar to part (ii), so we skip some details. Here also we consider two different cases.

Case 1: When E satisfies the condition P_m for $m \in \{2, 3, 4, 6\}$, by Theorem 4.3.4, the $\mathrm{GL}(2, K)$ representation is of the form $\pi_E = \chi \times \chi^{-1}$, where χ is a character of $K^\times$ such that χ^m is unramified. Also, $a(\chi) = 1$ if E satisfies P_2 or P_4, and $a(\chi) = 2$ if E satisfies P_3 or P_6. Then, by Table 3.2, $\mathrm{sym}^3(\pi_E) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$ and $a(\mathrm{sym}^3(\pi_E)) = 2a(\chi^3) + 2a(\chi)$. Now, using similar arguments as in case 1 of part (ii), we get $a(\mathrm{sym}^3(\pi_E)) = 4$ when E satisfies P_2 or P_4 or P_3, and $a(\mathrm{sym}^3(\pi_E)) = 6$ when E satisfies the condition P_6. Hence, we obtain the column of $a(\mathrm{sym}^3(\pi_E))$ in Table 5.5 for the case $\pi_E = \chi \times \chi^{-1}$.

Case 2: When E satisfies one of the conditions in $\{\mathrm{S}_4, \mathrm{S}_3, \mathrm{S}_6, \mathrm{S}_6', \mathrm{S}_6''\}$, by Theorem 4.3.9, the corresponding $\mathrm{GL}(2, K)$ representation is of the form $\pi_E = \omega_{F,\xi}$, where F is a quadratic extension of K and ξ is a character of $F^\times$ with $\xi^\sigma = \xi^{-1}$. Then, using Table 3.1, the local parameter of $\mathrm{sym}^3(\pi_E)$ is of the form $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi) \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ and $\mathrm{sym}^3(\pi_E) = \omega_{F,\xi} \oplus \omega_{F,\xi^3}$. Now, we consider two cases.

Let F is the unramified quadratic extension of K. Then, from Table 3.2 we get $a(\mathrm{sym}^3(\pi_E)) = 2a(\xi^3) + 2a(\xi)$. By Theorem 4.3.9, F/K is unramified when E satisfies the condition S_m for $m \in \{3, 4, 6\}$. In this case, the order of $\xi|_{\mathfrak{o}_F^\times}$ is m with $a(\xi) = 1$ for $m = 4$, and $a(\xi) = 2$ for $m = 3$ or 6. Then, using arguments similar to the case 2 of part (ii), one can easily verify that $a(\mathrm{sym}^3(\pi_E)) = 4$ when $m = 3$ or 4 and $a(\mathrm{sym}^3(\pi_E)) = 6$ when $m = 6$.

Suppose that F is a ramified quadratic extension of K. From Table 3.2 we get $a(\mathrm{sym}^3(\pi_E)) = a(\xi^3) + a(\xi) + 2$. Now, by Theorem 4.3.9, F/K is ramified when E satisfies either S_6' or S_6''. In both these cases, the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6. Also, $a(\xi) = 2$ if E satisfies the condition S_6' and $a(\xi) = 4$ if E satisfies the condition S_6''. Now, $0 < a(\xi^3) \leq a(\xi)$ and

$\xi|^3_{\mathfrak{o}_F^\times} \neq 1$. Again, we have $1+\mathfrak{p}_F \subset \mathfrak{o}_F^{\times 2}$. So, $\xi^3|_{1+\mathfrak{p}_F} = 1$, i.e., $a(\xi^3) = 1$. Hence we get $a(\mathrm{sym}^3(\pi_E)) = 5$ when E satisfies S_6' and $a(\mathrm{sym}^3(\pi_E)) = 7$ when E satisfies S_6''. Thus, we obtain the column of $a(\mathrm{sym}^3(\pi_E))$ in Table 5.5 when $\pi_E = \omega_{F,\xi}$. □

5.3 L-factor of $\mathrm{sym}^3(\pi_E)$

In this section, we find the L-factor $L(s, \mathrm{sym}^3(\pi_E))$ of $\mathrm{sym}^3(\pi_E)$ attached to an elliptic curve E/K in Tables 5.6 and 5.7. We again consider three cases here.

Case I: Assume that E has potential multiplicative reduction over any p-adic field K. Then we have $\pi_E = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GL}(2,K)}$ and $\mathrm{sym}^3(\pi_E) = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GSp}(4,K)}$, where $(\gamma(E/K), \cdot)$ is the quadratic character of $K^\times$ defined by the Hilbert symbol $(\cdot,\cdot)$. Then, by Table A.6 from [37], $L(s, \mathrm{sym}^3(\pi_E)) = L(s, \nu^{\frac{3}{2}}(\gamma(E/K), \cdot))$. Recall that ν is the normalized absolute value such that $\nu(\varpi_K) = q^{-1}$. For a character χ of $K^\times$, the symbol $L(s,\chi)$ is defined as follows

$$L(s,\chi) = \begin{cases} (1-\chi(\varpi_K)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases} \tag{5.3}$$

By Theorem 4.1.1, when E/K has split multiplicative reduction, $(\gamma(E/K), \cdot)$ is trivial and $(\gamma(E/K), \varpi_K) = 1$. When E/K has non-split multiplicative reduction, $(\gamma(E/K), \cdot)$ is the unique nontrivial and unramified character with $(\gamma(E/K), \varpi_K) = -1$ and when E/K has additive but potential multiplicative reduction, $(\gamma(E/K), \cdot)$ is ramified. So using (5.3) we get

$$L(s, \mathrm{sym}^3(\pi_E)) = \begin{cases} (1-q^{-s-\frac{3}{2}})^{-1} & \text{if } (\gamma(E/K),\cdot) \text{ is trivial,} \\ (1+q^{-s-\frac{3}{2}})^{-1} & \text{if } (\gamma(E/K),\cdot) \text{ is nontrivial and unramified,} \\ 1 & \text{if } (\gamma(E/K),\cdot) \text{ is ramified.} \end{cases}$$

Thus we obtain the first row of Table 5.6.

Case II: Assume that E has additive but potential good reduction over a p-adic field K with $p \geq 3$ and $\pi_E = \chi \times \chi^{-1}$. Then we have $\mathrm{sym}^3(\pi_E) = \chi^4 \times \chi^2 \rtimes \chi^{-3}$. Again, by Table A.6 from [37], $L(s, \mathrm{sym}^3(\pi_E)) = L(s,\chi)L(s,\chi^{-1})L(s,\chi^3)L(s,\chi^{-3})$. In this case, χ is always ramified. So, we get a non-trivial L-factor only when $a(\chi^3) = 0$ and in that case $L(s, \mathrm{sym}^3(\pi_E)) = L(s,\chi^3)L(s,\chi^{-3})$. Thus we obtain the second row of Table 5.6 and the rows of Table 5.7 when $\pi_E = \chi \times \chi^{-1}$.

Case III: Assume that E has additive but potential good reduction over a p-adic field K with $p \geq 3$ and $\pi_E = \omega_{F,\xi}$, where F is a quadratic extension of K and ξ is a character of $F^\times$ such that $\xi^\sigma = \xi^{-1}$. Then, using Table 3.1, the local parameter of $\mathrm{sym}^3(\pi_E)$ is of the form $\mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi) \oplus \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)$ and $\mathrm{sym}^3(\pi_E) = \omega_{F,\xi} \oplus \omega_{F,\xi^3}$. Using Property (L2) in Section 8 of [39], we get

$$L(s, \mathrm{sym}^3(\omega_{F,\xi})) = L(s, \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi))L(s, \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/K)}(\xi^3)) = L(s,\xi)L(s,\xi^3). \tag{5.4}$$

Since $a(\xi) \geq 1$ for all cases, from (5.4) it is clear that $L(s, \mathrm{sym}^3(\pi_E))$ is nontrivial if and only if ξ^3 is unramified. So, we get $L(s, \mathrm{sym}^3(\pi_E)) = 1$ for all cases in Tables 5.6 and 5.7 when $\pi_E = \omega_{F,\xi}$ except for the case when $\pi_E = \omega_{F,\xi}$ such that F/K is unramified and the order of $\xi|_{\mathfrak{o}_F^\times}$ is 3.

Suppose that $\pi_E = \omega_{F,\xi}$ such that F/K is unramified and the order of $\xi|_{\mathfrak{o}_F^\times}$ is 3. Since F/K is unramified, we have $\varpi_F = \varpi_K$, and $\xi(\varpi_F) = -1$. So, we get $\xi^3(\varpi_F) = -1$. Also, the order q_F of the residue field of F is equal to q^2 in this case. Hence, we get

$$L(s, \mathrm{sym}^3(\omega_{F,\xi})) = \begin{cases} (1 - \xi^3(\varpi_F)q_F^{-s})^{-1} = \frac{1}{1+q^{-2s}} & \text{if } \xi^3 \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Thus we obtain the last row of Table 5.6 and the rows of Table 5.7 when $\pi_E = \omega_{F,\xi}$.

Table 5.6: The spin L-factor of $\mathrm{sym}^3(\pi_E)$.

Here E has potential multiplicative reduction or E has additive but potential good reduction with $p \geq 5$. Here $\gamma = \gamma(E/K)$ is the γ-invariant of E, where $(\cdot,\cdot)$ is the Hilbert symbol. For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition on E/K	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Condition on π_E	$L(s,\mathrm{sym}^3(\pi_E))$
$j(E) \notin \mathfrak{o}_K$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GL}(2)}$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GSp}(4)}$	$(\gamma,\cdot)$ is ram.	1
			$(\gamma,\cdot)$ is trivial	$\frac{1}{1-q^{-3/2-s}}$
			$(\gamma,\cdot)$ is unr. nontrivial	$\frac{1}{1+q^{-3/2-s}}$
$j(E) \in \mathfrak{o}_K$ $(p-1)v_p(\Delta)\equiv 0 \bmod 12$	$\chi \times \chi^{-1}$ $a(\chi)=1$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right)=2$	1
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right)=3$	$L(s,\chi^3)L(s,\chi^{-3})$
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right)=4$	1
			$o\left(\chi\vert_{\mathfrak{o}_K^\times}\right)=6$	1
$j(E) \in \mathfrak{o}_K$ $(p-1)v_p(\Delta)\not\equiv 0 \bmod 12$	$\omega_{F,\xi}$ F/K is unr. $a(\xi)=1$ $\xi(\varpi_F)=-1$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right)=3$	$\frac{1}{1+q^{-2s}}$
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right)=4$	1
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\vert_{\mathfrak{o}_F^\times}\right)=6$	1

Table 5.7: The spin L-factor of $\mathrm{sym}^3(\pi_E)$ when E has additive but potential good reduction with $p = 3$.

For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition from Table 4.1	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Condition on π_E	$L(s,\mathrm{sym}^3(\pi))$
P_2	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 2$	1
P_3	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 2$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 3$	$L(s,\chi^3)L(s,\chi^{-3})$
P_4	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 4$	1
P_6	$\chi \times \chi^{-1}$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$	$a(\chi) = 2$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = 6$	1
S_4	$\omega_{F,\xi}$ $F=K(i)$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)	$a(\xi) = 1$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 4$	1
S_3	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 3$	$\frac{1}{1+q^{2s}}$
S_6	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (unramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	1
S_6'	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (ramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$a(\xi) = 2$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	1
S_6''	$\pi = \omega_{F,\xi}$ $F=K(\sqrt{\Delta})$ (ramified)	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$a(\xi) = 4$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	1

5.4 ε-factor of $\mathrm{sym}^3(\pi_E)$

In this section, we find the ε-factor $\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right)$ of $\mathrm{sym}^3(\pi_E)$ attached to an elliptic curve E/K in Tables 5.8 and 5.9. We again consider three cases here.

Case I: When $\pi_E = (\gamma(E/K),\cdot)\mathrm{St}_{\mathrm{GL}(2)}$, we have $\mathrm{sym}^3(\pi_E) = (\gamma(E/K),\cdot)\mathrm{St}_{\mathrm{GSp}(4)}$. Then, from Table A.9 in [37], we get

$$\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) = \begin{cases} 1 & \text{if } (\gamma(E/K),\cdot) \text{ is ram,} \\ -(\gamma(E/K),\varpi_K) & \text{if } (\gamma(E/K),\cdot) \text{ is unr.} \end{cases} \tag{5.5}$$

When $(\gamma(E/K),\cdot)$ is non-trivial and unramified, $(\gamma(E/K),\varpi_K) = -1$. Thus, we obtain the first row of Table 5.8.

Case II: When $\pi_E = \chi\times\chi^{-1}$ we have $\mathrm{sym}^3(\pi_E) = \chi^4\times\chi^2\rtimes\chi^{-3}$. Then, from Table A.9 in [37], we get $\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) = \chi^4(-1) = 1$. Thus, we obtain the second row of Table 5.8 and the first row of Table 5.9. **Case III:** Let $\pi_E = \omega_{F,\xi}$, where F is a quadratic character of K and ξ is a character of $F^\times$ such that $\xi^\sigma = \xi^{-1}$. So, $\mathrm{sym}^3(\pi_E) = \omega_{F,\xi}\oplus\omega_{F,\xi^3}$. Then, using the property (ϵ2) in Section 11 of [39], we get

$$\begin{aligned} \varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) &= \varepsilon\left(\frac{1}{2},\omega_{F,\xi}\right)\varepsilon\left(\frac{1}{2},\omega_{F,\xi^3}\right) \\ \text{and} \qquad \varepsilon\left(\frac{1}{2},\omega_{F,\xi}\right) &= \varepsilon\left(\frac{1}{2},\chi_{F/K},\psi\right)\varepsilon\left(\frac{1}{2},\xi,\psi\circ\mathrm{tr}\right). \end{aligned} \tag{5.6}$$

Here, $\chi_{F/K}$ is the quadratic character of $K^\times$ associated to the quadratic extension F/K that factors through the norm map $N_{F/K}$, tr is the trace map from F to K, and ψ is a non-trivial additive character of K with the conductor (exponent) $a(\psi)=0$. Calculation of $\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right)$ is a bit involved. We consider two separate cases.

<u>Case 1:</u> Suppose that F is the unramified quadratic extension of K. In this case, ξ^m is unramified for $m\in\{3,4,6\}$ and $\xi(\varpi_F) = -1$. Also, $\chi_{F/K}$ is the unramified quadratic

Table 5.8: The ε-factor of $\mathrm{sym}^3(\pi_E)$.

Here E has potential multiplicative reduction or E has additive but potential good reduction with $p \geq 5$. Here $\gamma = \gamma(E/K)$ is the γ-invariant of E, where $(\cdot,\cdot)$ is the Hilbert symbol and $e = \frac{12}{\gcd(12, v(\Delta))}$. For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition on E/K	$\mathrm{GL}(2,K)$ rep. π_E	$\mathrm{GSp}(4,K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Condition on π_E	$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right)$
$j(E) \notin \mathfrak{o}_K$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GL}(2)}$	$(\gamma,\cdot)\mathrm{St}_{\mathrm{GSp}(4)}$	$(\gamma,\cdot)$ is ram.	1
			$(\gamma,\cdot)$ is trivial	-1
			$(\gamma,\cdot)$ is unr. nontrivial	1
$j(E) \in \mathfrak{o}_K$ $(p-1)v_p(\Delta) \equiv 0 \mod 12$	$\chi \times \chi^{-1}$ $a(\chi) = 1$ $o\left(\chi\|_{\mathfrak{o}_K^\times}\right) = e$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$		1
$j(E) \in \mathfrak{o}_K$ $(p-1)v_p(\Delta) \not\equiv 0 \mod 12$	$\omega_{F,\xi}$ F/K is unr. $a(\xi)=1$ $\xi(\varpi_F)=-1$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 3$	-1
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)	$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 4$	1
		$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)	$o\left(\xi\|_{\mathfrak{o}_F^\times}\right) = 6$	1

Table 5.9: The ε-factor of $\mathrm{sym}^3(\pi_E)$ when E has additive but potential good reduction with $p = 3$.

Here, φ is a character of $K^\times$ such that $\xi^3 = \varphi \circ N_{F/K}$. For a character ψ, the symbol $o(\psi)$ denotes the order of ψ.

Condition from table 4.1	GL$(2, K)$ representation π_E	GSp$(4, K)$ L-packet $\mathrm{sym}^3(\pi_E)$	Extra condition on K	$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right)$
P_m $m \in \{2, 3, 4, 6\}$	$\chi \times \chi^{-1}$ $a(\chi) = 1$ $o\left(\chi\vert_{\mathfrak{o}_K^\times}\right) = m$	$\chi^4 \times \chi^2 \rtimes \chi^{-3}$		1
S_3	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 3$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)		1
S_4	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 1$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 4$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type VIII)		1
S_6	$\omega_{F,\xi}$ F/K is unr. $a(\xi) = 2$ $\xi(\varpi_F) = -1$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X)		-1
S_6'	$\omega_{F,\xi}$ F/K is ram. $a(\xi) = 2$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$-1 \in K^{\times 4}$	$\varepsilon(\frac{1}{2}, \pi_E)$
			$-1 \in K^{\times 2} \setminus K^{\times 4}$	$-\varepsilon(\frac{1}{2}, \pi_E)$
			$-1 \notin K^{\times 2}$	$\varepsilon(\frac{1}{2}, \pi_E)$
S_6''	$\omega_{F,\xi}$ F/K is ram. $a(\xi) = 4$ $o\left(\xi\vert_{\mathfrak{o}_F^\times}\right) = 6$	$\omega_{F,\xi} \oplus \omega_{F,\xi^3}$ (type X/s.c.)	$-1 \in K^{\times 4}$	$\varepsilon(\frac{1}{2}, \pi_E)$
			$-1 \in K^{\times 2} \setminus K^{\times 4}$	$-\varepsilon(\frac{1}{2}, \pi_E)$
			$-1 \notin K^{\times 2}$	$\varepsilon(\frac{1}{2}, \pi_E)$

character of $K^\times$ with $\chi_{F/K}(\varpi_K) = -1$. So, we get $\varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right) = 1$ and

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi}\right) = \varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right)\varepsilon\left(\frac{1}{2}, \xi, \psi \circ \mathrm{tr}\right) = \varepsilon\left(\frac{1}{2}, \xi, \psi \circ \mathrm{tr}\right). \tag{5.7}$$

The following result from [11] is useful for computing $\varepsilon\left(\frac{1}{2}, \xi, \psi \circ \mathrm{tr}\right)$.

Lemma 5.4.1. *Let F be a quadratic extension of K such that $F = K(\sqrt{u})$ with $u \in K^\times \setminus K^{\times 2}$. Let ϕ be any character of $F^\times$ such that $\phi|_{K^\times} = 1$. Then the root number*

$$W(\phi) = \varepsilon\left(\frac{1}{2}, \phi, \psi\right) = \phi(\sqrt{u}),$$

where ψ is a non-trivial additive character of F with the conductor (exponent) $a(\psi) = 0$.

Let χ be the unramified character of $F^\times = \langle \varpi_F \rangle \times \mathfrak{o}_F^\times$, such that $\chi(\varpi_F) = -1$. Note that, $\chi|_{K^\times}(\varpi_K) = -1$ (since $\varpi_F = \varpi_K$) and $\chi|_{\mathfrak{o}_K^\times} = 1$. Also, $\chi_{F/K}$ is the character of $K^\times$ such that $\chi_{F/K}|_{\mathfrak{o}_K^\times} = 1$ and $\chi_{F/K}(\varpi_K) = -1$. Since $K^\times = \langle \varpi_K \rangle \times \mathfrak{o}_K^\times$, so $\chi|_{K^\times} = \chi_{F/K}$. Now, since $\pi_E = \omega_{F,\xi}$ has trivial central character, by Remark 2.2.9, $(\xi|_{K^\times}).(\chi_{F/K}) = 1$. This implies $(\xi.\chi)|_{K^\times} = 1$. Also, $\psi \circ \mathrm{tr}$ is an additive character of F. Let $f(F/K)$ be the degree of the residue field extension, $e(F/K)$ be the ramification index and $d(F/K)$ be the valuation of the discriminant of the field extension F/K. We have $d(F/K) = 0$ if F/K is unramified, and $d(F/K) = 1$ if F/K is ramified. Then, by Lemma 2.3 in [41], the conductor (exponent) $a(\psi \circ \mathrm{tr})$ of $\psi \circ \mathrm{tr}$ is given by

$$a(\psi \circ \mathrm{tr}) = e(F/K)a(\psi) - f(F/K)^{-1}d(F/K) = 1 \cdot 0 - 2^{-1} \cdot 0 = 0. \tag{5.8}$$

Then, using Lemma 5.4.1, we get

$$\varepsilon\left(\frac{1}{2}, \xi \cdot \chi, \psi \circ \mathrm{tr}\right) = (\xi \cdot \chi)(\sqrt{u}) = \xi(\sqrt{u}) \cdot \chi(\sqrt{u}). \tag{5.9}$$

On the other hand, using (3.2.6.3) in [49] for the unramified character χ of $F^\times$, we get

$$\begin{aligned}\varepsilon\left(\frac{1}{2},\xi\cdot\chi,\psi\circ\mathrm{tr}\right)&=\chi(\varpi_F)^{a(\xi)}\varepsilon\left(\frac{1}{2},\xi,\psi\circ\mathrm{tr}\right)\\&=(-1)^{a(\xi)}\varepsilon\left(\frac{1}{2},\xi,\psi\circ\mathrm{tr}\right).\quad(\text{since }\chi(\varpi_F)=-1.)\end{aligned}\tag{5.10}$$

Hence, using (5.7), (5.9) and (5.10), we get

$$\varepsilon\left(\frac{1}{2},\pi_E\right)=\varepsilon\left(\frac{1}{2},\omega_{F,\xi}\right)=(-1)^{a(\xi)}\xi(\sqrt{u}).\tag{5.11}$$

Also, since $\xi|_{K^\times}.\chi_{F/K}=1$, we have $\xi^3|_{K^\times}.\chi_{F/K}^3=\xi^3|_{K^\times}.\chi_{F/K}=(\xi^3\cdot\chi)|_{K^\times}=1$. Then, using Lemma 5.4.1 again for $\xi^3.\chi$, we get

$$\varepsilon\left(\frac{1}{2},\xi^3.\chi,\psi\circ\mathrm{tr}\right)=(\xi^3.\chi)(\sqrt{u})=\xi^3(\sqrt{u}).\chi(\sqrt{u})=\xi^3(\sqrt{u}).\tag{5.12}$$

Also, using (3.2.6.3) in [49] again we obtain

$$\begin{aligned}\varepsilon\left(\frac{1}{2},\xi^3.\chi,\psi\circ\mathrm{tr}\right)&=\chi(\varpi_F)^{a(\xi^3)}\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right)\\&=(-1)^{a(\xi^3)}\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right).\quad(\text{since }\chi(\varpi_F)=-1.)\end{aligned}\tag{5.13}$$

Then using (5.12) and (5.13) we get

$$\begin{aligned}\varepsilon\left(\frac{1}{2},\omega_{F,\xi^3}\right)&=\varepsilon\left(\frac{1}{2},\chi_{F/K},\psi\right)\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right)\\&=\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right)\left(\text{since }\chi_{F/K}\text{ is unr., so }\varepsilon\left(\frac{1}{2},\chi_{F/K},\psi\right)=1\right)\\&=(-1)^{a(\xi^3)}\xi^3(\sqrt{u}).\end{aligned}\tag{5.14}$$

Hence, using (5.6), (5.11) and (5.14),

$$\begin{aligned}\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) &= ((-1)^{a(\xi)}\xi^3(\sqrt{u}))((-1)^{a(\xi^3)}\xi(\sqrt{u}))\\ &= (-1)^{a(\xi)+a(\xi^3)}\xi^4(\sqrt{u})\\ &= (-1)^{a(\xi)+a(\xi^3)}\xi^2(u) = (-1)^{a(\xi)+a(\xi^3)}.\end{aligned}\tag{5.15}$$

When K has residue characteristic $p \geq 5$, using case 2 of the proof of (ii) in Theorem 5.2.1, we get the following for $e = \frac{12}{\gcd(12,v(\Delta))} \in \{3,4,6\}$.

$$\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) = \begin{cases}1 & \text{when the order of } \xi \text{ is 4 or 6,}\\ -1 & \text{when the order of } \xi \text{ is 3.}\end{cases}\tag{5.16}$$

When K has residue characteristic 3, using case 2 of the proof of (iii) in Theorem 5.2.1, we get

$$\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right) = \begin{cases}1 & \text{when } E \text{ satisfies } \mathrm{S}_3 \text{ or } \mathrm{S}_4,\\ -1 & \text{when } E \text{ satisfies } \mathrm{S}_6.\end{cases}\tag{5.17}$$

Thus, we obtain the last row of Table 5.8 when $\pi_E = \omega_{F,\xi}$ and the rows of Table 5.9 when E satisfies S_m for $m \in \{3,4,6\}$.

Case 2: Let F be a ramified quadratic extension of K. In this case, K has residue characteristic 3 and E satisfies S_6' or S_6''. In this case, the order of $\xi|_{\mathfrak{o}_F^\times}$ is 6 and $\varpi_K = N_{F/K}(\varpi_F)$. Also, by (5.2) of Lemma 5.1.3, we have the following

$$\xi^6(\varpi_F) = \begin{cases}1 & \text{if } -1 \in K^{\times 2},\\ -1 & \text{if } -1 \notin K^{\times 2}.\end{cases}\tag{5.18}$$

So, $\xi^6 = 1$ on $F^\times$ if $-1 \in K^{\times 2}$ and $\xi^{12} = 1$ on $F^\times$ if $-1 \notin K^{\times 2}$. Now, we consider two scenarios.

First, let us assume $-1 \in K^{\times 2}$. From Table 5.2, if $-1 \in K^{\times 2}$, then $\mathrm{sym}^3(\pi_E)$ is of type X. Then $\mathrm{sym}^3(\pi_E)$ has L-parameter form $\varphi\chi_{F/K} \oplus \varphi\mu \oplus \varphi$, where $\varphi\mu = \mathrm{ind}_{W(\bar{K}/F)}^{W(\bar{K}/F)}(\xi)$ with an irreducible parameter μ of $\mathrm{GL}(2,K)$ and $\xi^3 = \varphi \circ N_{F/K}$. Now, $\varphi^2 = \xi|_{K^\times}^3 = \chi_{F/K}$. Note that, since $\varphi\chi_{F/K} = \varphi^{-1}$, we get the following (see Section 12 of [39])

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \varphi, \psi\right)\varepsilon\left(\frac{1}{2}, \varphi\chi_{F/K}, \psi\right) = \varphi(-1). \tag{5.19}$$

Then, using (5.6), we get (also see Table A.6 in [37])

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varphi(-1)\varepsilon\left(\frac{1}{2}, \pi_E\right). \tag{5.20}$$

Now, $-1 \in K^{\times 2}$, so $\varphi^2(-1) = \chi_{F/K}(-1) = 1$, i.e., $\varphi(-1) \in \{\pm 1\}$. Note that, $\varphi|_{\mathfrak{o}_K^\times}$ has order 4 and $\varphi|_{\mathfrak{o}_K^{\times 2}} \neq 1$. Now, $a(\mathrm{sym}^3(\omega_{F,\xi^3})) = a(\omega_{F,\xi}) + a(\omega_{F,\xi^3})$. Also, using the conductor formula for type X from Table A.9 in [37], $a(\mathrm{sym}^3(\omega_{F,\xi^3})) = a(\omega_{F,\xi}) + 2a(\varphi)$. So, $2a(\varphi) = a(\omega_{F,\xi^3}) = a(\xi^3) + 1 = 2$, i.e., $a(\varphi) = 1$. So, the induced character $\varphi : k^\times = \mathfrak{o}_K/(1+\mathfrak{p}) = \langle g \rangle \to \mathbb{C}^\times$ has order 4 such that $\varphi(g) = i$. The kernel of this map is $\langle g^4 \rangle$. Since $-1 \in k^{\times 2}$, so $-1 = g^{2m}$ for some $m \in \mathbb{N}$. Now if m is even, i.e., $-1 \in \langle g^4 \rangle$, then $\varphi(-1) = 1$. If m is odd, i.e., $-1 \in \langle g^2 \rangle \setminus \langle g^4 \rangle$, then $\varphi(-1) = -1$. Now, from Hensel's lemma it follows that $-1 \in k^{\times 4}$ if and only if $-1 \in K^{\times 4}$. So, we get $\varphi(-1) = 1$ if $-1 \in K^{\times 4}$ and $\varphi(-1) = -1$ if $-1 \in K^{\times 2} \setminus K^{\times 4}$.

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \begin{cases} \varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if } -1 \in K^{\times 4}, \\ -\varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if } -1 \in K^{\times 2} \setminus K^{\times 4}. \end{cases} \tag{5.21}$$

Next, let us assume $-1 \notin K^{\times 2}$. In this case, $\xi^{12} = 1$ on $F^\times$ and ω_{F,ξ^3} is a dihedral supercuspidal representation of trivial central character. Also, $a(\omega_{F,\xi^3}) = a(\xi^3) + 1 = 2$. Note that, $(\xi^3)^2$ is Galois invariant in this case, i.e., $\xi^6 = (\xi^6)^\sigma = (\xi^6)^{-1}$ on $F^\times$, here σ is the non-trivial element in $\mathrm{Gal}(F/K)$. So, by Corollary 3.2 in [45], ω_{F,ξ^3} is a triply imprimitive representation. Also, we use the following remark to show that such a representation exists and is unique when $-1 \notin K^{\times 2}$.

Remark 5.4.2. *$-1 \notin K^{\times 2}$ if and only if $f(K/\mathbb{Q}_3)$ is odd if and only if $q \equiv 3 \bmod 4$.*

Using Remark 5.4.2 and Theorem 4.1 from [45] for ω_{F,ξ^3}, we get that $\omega_{F,\xi^3} \cong \omega_{F',\mu}$, where F' is the unramified extension of K and μ is a character of $F'^{\times}$ with the following properties: $\mu(\varpi_{F'}) = -1$, $a(\mu) = 1$ and $\mu|_{\mathfrak{o}_{F'}}^{4} = 1$. Then, using some similar arguments from Case 1, we get the following formula as in (5.11):

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \omega_{F',\mu}\right) = (-1)^{a(\mu)}\mu(\sqrt{u}) = -\mu(\sqrt{u}), \tag{5.22}$$

where we are assuming $F' = K(\sqrt{u})$ for some $u \in K^{\times} \setminus K^{\times 2}$. Let f' be the residue field of F' and k be the residue field of K. Then $f'^{\times} \cong \mathbb{F}_{q^2}^{\times}$ and $k^{\times} \cong \mathbb{F}_q^{\times}$ are cyclic groups. Let $f'^{\times} = \langle g \rangle$. Note that, the action of the Frobenius automorphism on g^{q+1} is trivial since $(g^{q+1})^q = g^{q^2+q} = g^{q+1}$ and $g^{q^2-1} = (g^{q+1})^{q-1} = 1$, so we have $k^{\times} = \langle g^{q+1} \rangle$. Also, $g^{q+1} \notin k^{\times 2}$. Without loss of generality, we choose $u \in K^{\times}$ such that the residue class of u in $k^{\times}$ is g^{q+1} (it must be something in $k^{\times} \setminus k^{\times 2}$ since $u \in K^{\times} \setminus K^{\times 2}$). So, we may assume that, the residue class of $\sqrt{u} \in F'^{\times}$ is $g^{\frac{q+1}{2}} \in f'^{\times}$. By Theorem 4.1 from [45], the induced character $\bar{\mu}$ on $f'^{\times} = \mathfrak{o}_{F'}^{\times}/1 + \mathfrak{p}_{F'}$ is given by $\bar{\mu}(g) = i$, and we get the character μ on $F'^{\times}$ by inflating $\bar{\mu}$ to a character of $\mathfrak{o}_{F'}^{\times}$ and extending to $F'^{\times}$ by setting $\xi(\varpi_{F'}) = -1$. Using Remark 5.4.2, we get $\frac{q+1}{2} \equiv 2 \bmod 4$ and $\bar{\mu}(g^{\frac{q+1}{2}}) = i^{\frac{q+1}{2}} = -1$. So, we get $\mu(\sqrt{u}) = -1$ Hence, when $-1 \notin K^{\times 2}$, we get

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = 1 \text{ and } \varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varepsilon\left(\frac{1}{2}, \pi_E\right). \tag{5.23}$$

When is $\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\omega_{F,\xi})\right) = \varepsilon(\frac{1}{2}, \omega_{F,\xi})$ for F/K unramified?

Suppose that, the residual characteristic of K is ≥ 5 and the $\mathrm{GL}(2, K)$ representation π_E attached to an elliptic curve E/K is $\pi_E = \omega_{F,\xi}$, where F/K is an unramified quadratic extension and ξ is a character of $F^\times$. In this section we investigate the relationship between $\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\omega_{F,\xi})\right)$ and $\varepsilon\left(\frac{1}{2}, , \omega_{F,\xi}\right)$. Using (5.6) and (5.14), we have

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varepsilon\left(\frac{1}{2}, \omega_{F,\xi}\right)\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \omega_{F,\xi}\right)(-1)^{a(\xi^3)}\xi^3(\sqrt{u}), \tag{5.24}$$

where $F = K(\sqrt{u})$ with $u \in K^\times \setminus K^{\times 2}$ and ξ is a character on $F^\times$. When $\xi|_{\mathfrak{o}_F^\times}$ has order 3, from (5.24) we get

$$\boxed{\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varepsilon\left(\frac{1}{2}, \pi_E\right).} \tag{5.25}$$

When $\xi|_{\mathfrak{o}_F^\times}$ has order m for $m = 4$ or 6, $a(\xi^3) = 1$. Then, from (5.24) we get

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = -\varepsilon\left(\frac{1}{2}, \omega_{F,\xi}\right)\xi^3(\sqrt{u}). \tag{5.26}$$

Let k be the residue field of K of order q, i.e, $k \cong \mathbb{F}_q$. Then the residue field f of F is isomorphic to $\mathbb{F}_{q^2}$. Suppose that $\mathbb{F}_{q^2}^\times = \langle g \rangle$, then $k^\times = \mathbb{F}_q^\times = \langle g^{q+1} \rangle$. Without loss of generality, we choose $u \in K^\times$ such that the residue class of u in $k^\times$ is g^{q+1}. So, we may assume that, the residue class of $\sqrt{u} \in F^\times$ is $g^{\frac{q+1}{2}} \in f^\times$. Let $\bar{\xi}$ be the induced character on $f^\times$ and we get the character ξ on $F^\times$ by inflating $\bar{\xi}$ to a character of $\mathfrak{o}_F^\times$ and extending to $F^\times$ by setting $\xi(\varpi_F) = -1$.

When $\xi|_{\mathfrak{o}_F^\times}$ has order m for $m = 4$ or 6, the induced character $\bar{\xi}$ also has order m. There are exactly two characters of order m on $f^\times$ for $m = 4$ or 6 and $\bar{\xi}$ is given by

$\bar{\xi}(g) = e^{\frac{2\pi i}{m}}$ or $e^{-\frac{2\pi i}{m}}$. We assume that $\bar{\xi}(g) = e^{\frac{2\pi i}{m}}$. Then, we get

$$\bar{\xi}^3(g^{\frac{q+1}{2}}) = e^{\frac{6\pi i(q+1)}{2m}}. \tag{5.27}$$

Now, $\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = -\xi^3(\sqrt{u}) \in \{\pm 1\}$. So, $\xi^3(\sqrt{u}) \in \{\pm 1\}$. Also, note that, $\xi^3(\sqrt{u}) = 1$ if $\bar{\xi}^3(g^{\frac{q+1}{2}}) = 1$ and $\xi^3(\sqrt{u}) = -1$ if $\bar{\xi}^3(g^{\frac{q+1}{2}}) = -1$.

When $m = 6$, from (5.27) we get $\bar{\xi}^3(g^{\frac{q+1}{2}}) = 1$ if and only if $q + 1 \equiv 0 \bmod 4$ and $\bar{\xi}^3(g^{\frac{q+1}{2}}) = -1$ if and only if $q + 1 \equiv 2 \bmod 4$. So, using (5.26) and (5.27), we get

$$\xi^3(\sqrt{u}) = \begin{cases} 1 & \text{if} \quad q \equiv 3 \bmod 4, \\ -1 & \text{if} \quad q \equiv 1 \bmod 4. \end{cases}$$

and

$$\boxed{\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \begin{cases} -\varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if} \quad q \equiv 3 \bmod 4, \\ \varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if} \quad q \equiv 1 \bmod 4. \end{cases}} \tag{5.28}$$

When $m = 4$, we have $\bar{\xi}^3(g^{\frac{q+1}{2}}) = i^{\frac{3(q+1)}{2}} = (-i)^{\frac{q+1}{2}}$. Then, by (5.26) and (5.27), we get

$$\xi^3(\sqrt{u}) = \begin{cases} -1 & \text{if} \quad q \equiv 3 \bmod 8, \\ 1 & \text{if} \quad q \equiv 7 \bmod 8. \end{cases}$$

and

$$\boxed{\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \begin{cases} \varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if} \quad q \equiv 3 \bmod 8, \\ -\varepsilon\left(\frac{1}{2}, \pi_E\right) & \text{if} \quad q \equiv 7 \bmod 8. \end{cases}} \tag{5.29}$$

Remark 5.4.3. *Suppose that the residual characteristic of K is 3 and $v(3) = 1$. If an elliptic curve E/K satisfies the condition* S_3 *from Table 4.1, then the* $\mathrm{GL}(2, K)$ *representation π_E attached to E satisfies* (5.25)*. If an elliptic curve E/K satisfies the condition*

S_4 *from Table 4.1, then the* $\mathrm{GL}(2,K)$ *representation* π_E *attached to* E *satisfies* (5.29). *One can show this using similar arguments as in the proofs of* (5.25) *and* (5.29).

Another way of computing $\varepsilon(\frac{1}{2}, \mathrm{sym}^3(\omega_{F,\xi}))$ when $F/\mathbb{Q}_3$ is ramified

Lemma 5.4.4. *Suppose that* $E/\mathbb{Q}_3$ *satisfies the condition* S_6'' *from Table 4.1. Then the* $\mathrm{GL}(2,\mathbb{Q}_3)$ *representation* π_E *associated to* E *is* $\pi_E = \omega_{F,\xi}$ *such that*

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varepsilon\left(\frac{1}{2}, \pi_E\right).$$

Proof. If E satisfies S_6'', then $\pi_E = \omega_{F,\xi}$ with $F \cong \mathbb{Q}_3(\sqrt{-3})$ by Lemma 4.3.17. We have

$$\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\omega_{F,\xi})\right) = \varepsilon\left(\frac{1}{2}, \omega_{F,\xi}\right)\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right), \tag{5.30}$$

where

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right)\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ \mathrm{tr}\right). \tag{5.31}$$

Here, $\psi : \mathbb{Q}_3 \to \mathbb{C}^\times$ is an additive character defined by $x \mapsto e^{2\pi i\{x\}_3}$ with conductor (exponent) 0, i.e., $\psi|_{\mathbb{Z}_3} = 1$ and $\{x\}_3$ denotes the 3-adic fractional part of $x \in \mathbb{Q}_3$. For any $x \in \mathbb{Q}_3$, we have $\{x\}_3 = \frac{m}{3^n} \in [0,1)$ for some m satisfying $0 \le m \le p^n - 1$ and $x - \{x\}_3 \in \mathbb{Z}_3$.

Also, we have $\xi|_{\mathbb{Q}_3^\times} \cdot \chi_{F/\mathbb{Q}_3} = 1$, where $\chi_{F/\mathbb{Q}_3}$ is a ramified quadratic character of $\mathbb{Q}_3^\times$. Here, we assume $\varpi_F = \sqrt{-3}$. By corollary 3 of section V3 of [46], we get $\chi_{F/\mathbb{Q}_3}(1 + 3\mathbb{Z}_3) = \chi_{F/K}\left(N(1+\mathfrak{p}_F^2)\right) \Rightarrow \chi_{F/\mathbb{Q}_3}|_{1+3\mathbb{Z}_3} = 1$. Then we have the following (see (ϵ3) from

Section 11 in [39])

$$\begin{aligned}
\varepsilon\left(\frac{1}{2},\chi_{F/\mathbb{Q}_3},\psi\right) &= 3^{-\frac{a\left(\chi_{F/\mathbb{Q}_3}\right)}{2}}\int_{3^{-a\left(\chi_{F/\mathbb{Q}_3}\right)}\mathbb{Z}_3^\times}\chi_{F/\mathbb{Q}_3}^{-1}(x)\psi(x)dx\\
&= 3^{-\frac{1}{2}}\int_{3^{-1}\mathbb{Z}_3^\times}\chi_{F/\mathbb{Q}_3}(x)\psi(x)dx \quad \text{(since } \chi_{F/\mathbb{Q}_3} \text{ is quadratic.)}\\
&= 3^{-\frac{1}{2}}\left[\int_{3^{-1}(1+3\mathbb{Z}_3)}\chi_{F/\mathbb{Q}_3}(x)\psi(x)dx+\int_{3^{-1}(2+3\mathbb{Z}_3)}\chi_{F/\mathbb{Q}_3}(x)\psi(x)dx\right]\\
&= 3^{-\frac{1}{2}}\left[\chi_{F/\mathbb{Q}_3}(3^{-1})\int_{(3^{-1}+\mathbb{Z}_3)}\psi(x)dx+\chi_{F/\mathbb{Q}_3}(2\cdot 3^{-1})\int_{(2\cdot 3^{-1}+\mathbb{Z}_3)}\psi(x)dx\right]\\
&= 3^{-\frac{1}{2}}\chi_{F/\mathbb{Q}_3}(3)\left[\psi\left(\frac{1}{3}\right)\int_{\mathbb{Z}_3}\psi(x)dx+\chi_{F/\mathbb{Q}_3}(2)\psi\left(\frac{2}{3}\right)\int_{\mathbb{Z}_3}\psi(x)dx\right].
\end{aligned}$$

Now, $\chi_{F/\mathbb{Q}_3}(3^{-1})=\chi_{F/\mathbb{Q}_3}(3)=\chi_{F/\mathbb{Q}_3}(N(\sqrt{-3}))=1$ and $\chi_{F/\mathbb{Q}_3}(2)=\chi_{F/\mathbb{Q}_3}(-1)=-1$ (since $-1\in W_2$ and $\chi_{F/\mathbb{Q}_3}|_{W_2}\neq 1$). Also, $\psi|_{\mathbb{Z}_3}=1$ and dx is the Haar measure giving $\mathbb{Z}_3$ volume 1. So we get,

$$\begin{aligned}
\varepsilon\left(\frac{1}{2},\chi_{F/\mathbb{Q}_3},\psi\right) &= 3^{-\frac{1}{2}}\left[\psi\left(\frac{1}{3}\right)\int_{\mathbb{Z}_3}dx-\psi\left(\frac{2}{3}\right)\int_{\mathbb{Z}_3}dx\right]\\
&= 3^{-\frac{1}{2}}\left[\psi\left(\frac{1}{3}\right)-\psi\left(\frac{2}{3}\right)\right]=3^{-\frac{1}{2}}\left[e^{\frac{2\pi i}{3}}-e^{\frac{4\pi i}{3}}\right]=i.
\end{aligned}\tag{5.32}$$

Now, we calculate $\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right)$. Here, $\psi\circ\mathrm{tr}$ is an additive character of F. By Lemma 2.3 in [41], with the conductor (exponent) of $\psi\circ\mathrm{tr}$ is given by

$$a(\psi\circ\mathrm{tr})=e(F/\mathbb{Q}_3)a(\psi)-f(F/\mathbb{Q}_3)^{-1}d(F/\mathbb{Q}_3)=2\cdot 0-1\cdot 1=-1,$$

i.e., $(\psi\circ\mathrm{tr})|_{\varpi_F^{-1}\mathfrak{o}_F}=1$. Since $a(\psi\circ\mathrm{tr})\neq 0$, we can not calculate $\varepsilon\left(\frac{1}{2},\xi,\psi\circ\mathrm{tr}\right)$ directly. We need to find a character ψ_F of F such that $\psi_F^c=\psi\circ\mathrm{tr}$ for some $c\in F^\times$ with $a(\psi_F)=0$, i.e., we need to find an appropriate $c\in F^\times$ such that $\psi_F=(\psi\circ\mathrm{tr})^{c^{-1}}$ has conductor 0. Here, $(\psi\circ\mathrm{tr})^{c^{-1}}$ is defined as $(\psi\circ\mathrm{tr})^{c^{-1}}(x):=(\psi\circ\mathrm{tr})(c^{-1}x)$. We choose $c=\varpi_F$. Then $\psi_F=(\psi\circ\mathrm{tr})^{\varpi_F^{-1}}$ has conductor 0, since $\psi_F(x)=(\psi\circ\mathrm{tr})(\varpi_F^{-1}x)=1$ for all $x \in \mathfrak{o}_F$. So, using Proposition (i) in Section 11 of [39] we get

$$\varepsilon\left(\frac{1}{2}, \xi^3, \psi_F\right) = \varepsilon\left(\frac{1}{2}, \xi^3, (\psi \circ \mathrm{tr})^{\varpi_F^{-1}}\right) = \xi^3(\varpi_F^{-1})\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right).$$

So, $\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right) = \xi^3(\varpi_F)\varepsilon\left(\frac{1}{2}, \xi^3, \psi_F\right) = \xi^3(\varpi_F) \cdot \mathfrak{q}_F^{-\frac{a\left(\xi^3\right)}{2}} \int\limits_{\varpi_F^{-a(\xi^3)}\mathfrak{o}_F^\times} \xi^{-3}(x)\psi_F(x)dx.$

We have $\mathfrak{q}_F = 3$. Also, since $a(\xi^3) = 1$ and $\mathfrak{o}_F^\times/(1+\mathfrak{p}_F) \cong \mathbb{Z}/2\mathbb{Z}$, we can write $\mathfrak{o}_F^\times = (1+\mathfrak{p}_F)\bigsqcup 2(1+\mathfrak{p}_F)$. Then we get

$$\begin{aligned}
\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right) &= \xi^3(\varpi_F) \cdot 3^{-\frac{1}{2}} \int\limits_{\varpi_F^{-1}(1+\mathfrak{p}_F)} \xi^{-3}(x)(\psi \circ \mathrm{tr})(\varpi_F^{-1}x)dx \\
&+ \xi^3(\varpi_F) \cdot 3^{-\frac{1}{2}} \int\limits_{\varpi_F^{-1}\cdot 2(1+\mathfrak{p}_F)} \xi^{-3}(x)(\psi \circ \mathrm{tr})(\varpi_F^{-1}x)dx \\
&= \xi^3(\varpi_F) \cdot 3^{-\frac{1}{2}}\xi^{-3}(\varpi_F^{-1}) \int\limits_{\varpi_F^{-1}+\mathfrak{o}_F} (\psi \circ \mathrm{tr})(\varpi_F^{-1}x)dx \\
&+ \xi^3(\varpi_F) \cdot 3^{-\frac{1}{2}}\xi^{-3}(2\varpi_F^{-1}) \int\limits_{2\varpi_F^{-1}+\mathfrak{o}_F} (\psi \circ \mathrm{tr})(\varpi_F^{-1}x)dx
\end{aligned}$$

$$\begin{aligned}
&= \xi^6(\varpi_F) \cdot 3^{-\frac{1}{2}} \left((\psi \circ \mathrm{tr})(\varpi_F^{-2}) \int\limits_{\mathfrak{o}_F} dx + \xi^{-3}(2)(\psi \circ \mathrm{tr})(2\varpi_F^{-2}) \int\limits_{\mathfrak{o}_F} dx\right) \\
&= \xi^6(\varpi_F) \cdot 3^{-\frac{1}{2}} \left((\psi \circ \mathrm{tr})(\varpi_F^{-2}) - (\psi \circ \mathrm{tr})(2\varpi_F^{-2})\right) \text{ (Since, } \xi^3(2) = -1) \\
&= \xi^6(\varpi_F) \cdot 3^{-\frac{1}{2}} \left(\psi(2\varpi_F^{-2}) - \psi(4\varpi_F^{-2})\right) \quad (\mathrm{tr}(a\varpi_F^{-2}) = 2a\varpi_F^{-2}.) \\
&= \xi^6(\varpi_F) \cdot 3^{-\frac{1}{2}} \left(\psi\left(-\frac{2}{3}\right) - \psi\left(-\frac{4}{3}\right)\right) \\
&= -3^{-\frac{1}{2}} \left(e^{\frac{-4\pi i}{3}} - e^{\frac{-8\pi i}{3}}\right) = -i. \quad (\xi(\varpi_F^2) = \xi(-3) = \chi_{F/\mathbb{Q}_3}(-3) = -1.)
\end{aligned}$$

Hence we get,

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right)\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right) = i \cdot (-i) = 1, \tag{5.33}$$

and

$$\boxed{\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right)=\varepsilon\left(\frac{1}{2},\pi_E\right).} \tag{5.34}$$

□

Lemma 5.4.5. *Let $E/\mathbb{Q}_3$ satisfies the condition S_6' from Table 4.1. Then the* $\mathrm{GL}(2,\mathbb{Q}_3)$ *representation π_E associated to E is $\pi_E=\omega_{F,\xi}$ such that*

$$F=\mathbb{Q}_3(\sqrt{\Delta})=\mathbb{Q}_3\left(\sqrt{\left(\frac{\Delta'}{3}\right)3}\right)\ \textit{where}\ \Delta'=\frac{\Delta}{3^{v_3(\Delta)}}, \tag{5.35}$$

and

$$\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\pi_E)\right)=\varepsilon\left(\frac{1}{2},\pi_E\right).$$

Proof. If $E/\mathbb{Q}_3$ satisfies S_6', then $\pi_E=\omega_{F,\xi}$, where $F=\mathbb{Q}_3(\sqrt{\Delta})$ is a ramified quadratic extension of $\mathbb{Q}_3$. To see that $F\cong\mathbb{Q}_3(\sqrt{-3})$ or $\mathbb{Q}_3(\sqrt{3})$ as in (5.35), one can use similar arguments as in Lemma 4.3.17. One need to consider all the possible values of $(v_3(\Delta),v_3(c_4),v_3(c_6))$ from Table 4.1 when E satisfies S_6' and use the relation $\Delta=\frac{c_4^3-c_6^2}{1728}$ to show that $3^{-v_3(\Delta)}\Delta\equiv -1$ or $1 \bmod 3$. In this case, $a(\xi)=2$, $\xi|_{\mathfrak{o}_F^\times}^6=1$, and $a(\xi^3)=1$. Again, $\varepsilon\left(\frac{1}{2},\mathrm{sym}^3(\omega_{F,\xi})\right)=\varepsilon\left(\frac{1}{2},\omega_{F,\xi}\right)\varepsilon\left(\frac{1}{2},\omega_{F,\xi^3}\right)$, where

$$\varepsilon\left(\frac{1}{2},\omega_{F,\xi^3}\right)=\varepsilon\left(\frac{1}{2},\chi_{F/K},\psi\right)\varepsilon\left(\frac{1}{2},\xi^3,\psi\circ\mathrm{tr}\right). \tag{5.36}$$

Here, $\psi:\mathbb{Q}_3\to\mathbb{C}^\times$ is an additive character defined by $x\mapsto e^{2\pi i\{x\}_3}$ with conductor (exponent) 0, i.e., $\psi|_{\mathbb{Z}_3}=1$. As before, $\{x\}_3$ denotes the 3-adic fractional part of $x\in\mathbb{Q}_3$. Also, $\chi_{F/\mathbb{Q}_3}$ is a ramified quadratic character of $\mathbb{Q}_3^\times$ with $\chi_{F/\mathbb{Q}_3}|_{1+3\mathbb{Z}_3}=1$. Now, we compute $\varepsilon\left(\frac{1}{2},\omega_{F,\xi^3}\right)$ in two cases.

Case 1: $F\cong\mathbb{Q}_3(\sqrt{-3})$

We assume that $\varpi_F=\sqrt{-3}$. So, $N_{F/\mathbb{Q}_3}(\varpi_F)=3$. Then, using a similar argument as in

(5.32), we get

$$\varepsilon\left(\frac{1}{2}, \chi_{F/\mathbb{Q}_3}, \psi\right) = i.$$

Now, we calculate $\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right)$ where ξ is any character of $F^\times$. We have $a(\psi\circ\mathrm{tr}) = -1$. Now, $\psi_F = (\psi\circ\mathrm{tr})^{\varpi_F^{-1}}$ has conductor 0, since $\psi_F(x) = (\psi\circ\mathrm{tr})(\varpi_F^{-1}x) = 1$ for all $x \in \mathfrak{o}_F$. Using Proposition (i) in Section 11 of [39], we get

$$\varepsilon\left(\frac{1}{2}, \xi^3, \psi_F\right) = \varepsilon\left(\frac{1}{2}, \xi^3, (\psi\circ\mathrm{tr})^{\varpi_F^{-1}}\right) = \xi^3(\varpi_F^{-1})\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right). \tag{5.37}$$

Then, we get

$$\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right) = \xi^3(\varpi_F)\varepsilon\left(\frac{1}{2}, \xi^3, \psi_F\right) = \xi^3(\varpi_F)\cdot\mathfrak{q}_F^{-\frac{a\left(\xi^3\right)}{2}} \int\limits_{\varpi_F^{-a(\xi^3)}\mathfrak{o}_F^\times} \xi^{-3}(x)\psi_F(x)dx.$$

Since $a(\xi^3) = 1$ and $\mathfrak{o}_F^\times/1+\mathfrak{p}_F \cong \mathbb{Z}/2\mathbb{Z}$, we can write $\mathfrak{o}_F^\times = (1+\mathfrak{p}_F)\bigsqcup 2(1+\mathfrak{p}_F)$. Then, we get the following

$$\begin{aligned}
\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right) &= \xi^3(\varpi_F)\cdot 3^{-\frac{1}{2}} \int\limits_{\varpi_F^{-1}(1+\mathfrak{p}_F)} \xi^{-3}(x)(\psi\circ\mathrm{tr})(\varpi_F^{-1}x)dx \\
&\quad + \xi^3(\varpi_F)\cdot 3^{-\frac{1}{2}} \int\limits_{\varpi_F^{-1}\cdot 2(1+\mathfrak{p}_F)} \xi^{-3}(x)(\psi\circ\mathrm{tr})(\varpi_F^{-1}x)dx \\
&= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left((\psi\circ\mathrm{tr})(\varpi_F^{-2})\int\limits_{\mathfrak{o}_F} dx + \xi^{-3}(2)(\psi\circ\mathrm{tr})(2\varpi_F^{-2})\int\limits_{\mathfrak{o}_F} dx\right) \\
&= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left(\psi(\mathrm{tr}(\varpi_F^{-2})) - \psi(\mathrm{tr}(2\varpi_F^{-2}))\right) \;\; (\chi_{F/\mathbb{Q}_3}(2) = \xi^3(2) = -1.) \\
&= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left(\psi(2\varpi_F^{-2}) - \psi(4\varpi_F^{-2})\right) \quad (\text{since } \mathrm{tr}(a\varpi_F^{-2}) = 2a\varpi_F^{-2}.) \\
&= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left(\psi\left(-\frac{2}{3}\right) - \psi\left(-\frac{4}{3}\right)\right).
\end{aligned}$$

Since $\xi(\varpi_F^2) = \xi(3) = -1$, we get $\xi(\varpi_F) \in \{\pm i\}$. So, $\xi^6(\varpi_F) = -1$. Then

$$\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right) = -3^{-\frac{1}{2}}\left(e^{\frac{-4\pi i}{3}} - e^{\frac{-8\pi i}{3}}\right) = -i.$$

So, we get

$$\varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right)\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right) = i \cdot (-i) = 1. \tag{5.38}$$

Case 2: $F \cong \mathbb{Q}_3(\sqrt{3})$

In this case, we assume $\varpi_F = \sqrt{3}$ and $N(\varpi_F) = -3$. Then

$$\begin{aligned}\varepsilon\left(\frac{1}{2}, \chi_{F/\mathbb{Q}_3}, \psi\right) &= 3^{-\frac{a\left(\chi_{F/\mathbb{Q}_3}\right)}{2}} \int_{3^{-a\left(\chi_{F/\mathbb{Q}_3}\right)}\mathbb{Z}_3^\times} \chi_{F/\mathbb{Q}_3}^{-1}(x)\psi(x)dx \\ &= 3^{-\frac{1}{2}} \int_{3^{-1}\mathbb{Z}_3^\times} \chi_{F/\mathbb{Q}_3}(x)\psi(x)dx \quad \text{(since } \chi_{F/\mathbb{Q}_3} \text{ is quadratic.)} \\ &= 3^{-\frac{1}{2}}\left[\int_{3^{-1}(1+3\mathbb{Z}_3)} \chi_{F/\mathbb{Q}_3}(x)\psi(x)dx + \int_{3^{-1}(2+3\mathbb{Z}_3)} \chi_{F/\mathbb{Q}_3}(x)\psi(x)dx\right]\end{aligned}$$

$$\begin{aligned}&= 3^{-\frac{1}{2}}\left[\chi_{F/\mathbb{Q}_3}(3^{-1})\int_{(3^{-1}+\mathbb{Z}_3)} \psi(x)dx + \chi_{F/\mathbb{Q}_3}(2\cdot 3^{-1})\int_{(2\cdot 3^{-1}+\mathbb{Z}_3)} \psi(x)dx\right] \\ &= 3^{-\frac{1}{2}}\chi_{F/\mathbb{Q}_3}(3)\left[\psi\left(\frac{1}{3}\right)\int_{\mathbb{Z}_3} \psi(x)dx + \chi_{F/\mathbb{Q}_3}(2)\psi\left(\frac{2}{3}\right)\int_{\mathbb{Z}_3} \psi(x)dx\right].\end{aligned}$$

Now, $\chi_{F/\mathbb{Q}_3}(-3) = \chi_{F/\mathbb{Q}_3}(N(\sqrt{3})) = 1$. So, $\chi_{F/\mathbb{Q}_3}(3) = \chi_{F/\mathbb{Q}_3}(-3)\chi_{F/\mathbb{Q}_3}(-1) = -1$. Also, $\chi_{F/\mathbb{Q}_3}(2) = \chi_{F/\mathbb{Q}_3}(-1) = -1$ (since $-1 \in W_2$ and $\chi_{F/\mathbb{Q}_3}|_{W_2} \neq 1$). So, we get

$$\begin{aligned}\varepsilon\left(\frac{1}{2}, \chi_{F/\mathbb{Q}_3}, \psi\right) &= 3^{-\frac{1}{2}}\left[-\psi\left(\frac{1}{3}\right)\int_{\mathbb{Z}_3} dx + \psi\left(\frac{2}{3}\right)\int_{\mathbb{Z}_3} dx\right] \\ &= 3^{-\frac{1}{2}}\left[-e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}}\right] = -i.\end{aligned}$$

Now, we calculate $\varepsilon\left(\frac{1}{2}, \xi^3, \psi \circ \mathrm{tr}\right)$ in a similar way as in case 1. But, we need to be careful about the fact that $\varpi_F = \sqrt{3}$ in this case. Using similar arguments as in case 1, we get

$$\begin{aligned}
\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right) &= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left(\psi(2\varpi_F^{-2}) - \psi(4\varpi_F^{-2})\right) \\
&= \xi^6(\varpi_F)\cdot 3^{-\frac{1}{2}}\left(\psi\left(\frac{2}{3}\right) - \psi\left(\frac{4}{3}\right)\right) \text{(this is different from case 1.)} \\
&= -3^{-\frac{1}{2}}\left(e^{\frac{4\pi i}{3}} - e^{\frac{8\pi i}{3}}\right) = i \quad \text{(since } \xi(\varpi_F)\in\{\pm i\} \Rightarrow \xi^6(\varpi_F) = -1.\text{)}
\end{aligned}$$

$$\Rightarrow \varepsilon\left(\frac{1}{2}, \omega_{F,\xi^3}\right) = \varepsilon\left(\frac{1}{2}, \chi_{F/K}, \psi\right)\varepsilon\left(\frac{1}{2}, \xi^3, \psi\circ\mathrm{tr}\right) = (-i)\cdot i = 1.$$

Hence, in both of the cases we get,

$$\boxed{\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_E)\right) = \varepsilon\left(\frac{1}{2}, \pi_E\right).} \tag{5.39}$$

□

When E satisfies S_6' or S_6'', one can also compute the epsilon factor $\varepsilon(\frac{1}{2}, \omega_{F,\xi})$. It turns out that $\varepsilon(\frac{1}{2}, \omega_{F,\xi})$ depends on the choice of $\xi(\varpi_F)$, and it can be either 1 or -1.

Chapter 6

Main global results

In this chapter, we discuss the global results on the sym^3 lifting of non-CM elliptic curves over $\mathbb{Q}$. These results are special and more precise versions of Theorem 3.2.1 and Corollary 3.5.5 for non-CM elliptic curves. Let E be a non-CM elliptic curve over $\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$. The discriminant Δ, the j-invariant $j(E)$, and the quantities c_4, c_6 are the usual constants given by (2.72). We can consider the same elliptic curve E over $\mathbb{Q}_p$ for each prime p. Let $\pi \cong \otimes_p \pi_p$ be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ with trivial central character attached to $E/\mathbb{Q}$. So, π_p is an irreducible admissible representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ associated with $E/\mathbb{Q}_p$.

Using our discussion on local presentations π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ in Chapter 4 and results on the L-packet $\mathrm{sym}^3(\pi_p)$ of $\mathrm{GSp}(4, \mathbb{Q}_p)$ from Chapter 5, we study the representation $\mathrm{sym}^3(\pi)$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ and the associated paramodular form f in this chapter. Here we have the following diagram:

$$\begin{tikzcd}
E/\mathbb{Q}_p \arrow[d] & E/\mathbb{Q} \text{ of conductor } N \arrow[l] \arrow[d] \arrow[r, squiggly, "\mathrm{sym}^3"] & f \in S_3(K(M)) \\
\pi_p \arrow[r] & \begin{array}{c} \pi \cong \bigotimes_p \pi_p \\ \text{on } \mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}}) \end{array} \arrow[r] & \begin{array}{c} \mathrm{sym}^3(\pi_p) \cong \bigotimes_p \mathrm{sym}^3(\pi_p) \\ \text{on } \mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}}) \end{array} \arrow[u]
\end{tikzcd}$$

Figure 6.1: The sym^3 lifting from elliptic curves to paramodular forms.

6.1 Paramodular level of $\mathrm{sym}^3(\pi)$ attached to an elliptic curve over $\mathbb{Q}$

Given the global minimal Weierstrass equation of a non-CM elliptic curve $E/\mathbb{Q}$ with good or potential multiplicative reduction at $p = 2$, the following theorem gives an algorithm for calculating the conductor N of E and the conductor of the automorphic representation $\mathrm{sym}^3(\pi)$, where π is the representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ attached to E.

Theorem 6.1.1. *Let $\pi \cong \otimes_p \pi_p$ be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ with trivial central character attached to a non-CM elliptic curve $E/\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$. So, the conductor $a(\pi)$ of π equals the conductor N of E. Suppose that E has good or potential multiplicative reduction at $p = 2$. Then there exists a cuspidal automorphic representation $\Pi \cong \otimes_p \Pi_p$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character which is unramified at each prime p not dividing N, such that*

(i) Π_p is a generic representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$ at each finite prime p.

(ii) Π_∞ is a holomorphic discrete series representation with minimal K-type $(3, 3)$.

(iii) $L(s, \Pi) = L(s, \pi, \mathrm{sym}^3) = L(s, E, \mathrm{sym}^3)$.

Moreover, the conductor of E and the conductor $a(\Pi)$ of Π are as follows

$$N = a(\pi) = \prod_{p|\Delta} p^i \quad \text{and} \quad a(\Pi) = \prod_{p|\Delta} p^k, \tag{6.1}$$

where the values of i, k are given in Table 6.1 for each p dividing the discriminant Δ of (2.71).

Proof. Most of the statements of the theorem including (i), (ii) follow from Theorem 3.2.1. Note that Π_∞ is the holomorphic discrete series representation of $\mathrm{GSp}(4, \mathbb{R})$

with minimal K-type $\Lambda = (3,3)$ as in the shaded region of Figure 2.1. Also, as a part of the correspondence between $E/\mathbb{Q}$ and π, we have $L(s,\pi) = L(s,E)$ (see (2.62)). So, it follows that $L(s,\pi,\mathrm{sym}^3) = L(s,E,\mathrm{sym}^3)$. Then, by (iii) of Theorem 3.2.1, we get $L(s,\Pi) = L(s,\pi,\mathrm{sym}^3) = L(s,E,\mathrm{sym}^3)$. Now, we only need to show that the conductor N of E and the conductor $a(\Pi)$ of $\Pi = \mathrm{sym}^3(\pi)$ are given by (6.1). Since $N = a(\pi) = \otimes_p p^{a(\pi_p)}$ and $a(\Pi) = \otimes_p p^{a(\Pi_p)}$, considering all possible values of $a(\pi_p)$ and $a(\Pi_p) = a(\mathrm{sym}^3(\pi_p))$ from Tables 5.3, 5.4, and 5.5, we see that N and $a(\Pi)$ are given by (6.1) where i, k are given in Table 6.1. This concludes the proof. $\square$

Furthermore, if the conductor of a non-CM elliptic curve $E/\mathbb{Q}$ is given along with its Weierstrass equation, then we have the following refinement of Theorem 6.1.1.

Corollary 6.1.2. *Let $\pi \cong \otimes_p \pi_p$ be the cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ attached to a non-CM elliptic curve $E/\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$ and the conductor N. Suppose that E has good or multiplicative reduction at $p = 2$. Then $\Pi = \mathrm{sym}^3(\pi)$ is a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$ with trivial central character, which is unramified at each prime p not dividing N and satisfies the properties (i), (ii), (iii) as in Theorem 6.1.1. Moreover, the conductor $a\left(\mathrm{sym}^3(\pi)\right)$ of $\mathrm{sym}^3(\pi)$ is given by*

$$a\left(\mathrm{sym}^3(\pi)\right) = N \prod_{\substack{p \mid N \\ v_p(\Delta) \not\equiv 0 \bmod 4}} p^2. \tag{6.2}$$

Proof. Here, we only need to prove the formula (6.2) for $a\left(\mathrm{sym}^3(\pi)\right)$. Note that, when E has good or multiplicative reduction at $p = 2$, (6.2) is a consequence of the following facts from Table 6.1.

$$\begin{aligned} k = i \quad &\text{if and only if} \quad v_p(\Delta) \equiv 0 \bmod 4. \\ k = i+2 \quad &\text{if and only if} \quad v_p(\Delta) \not\equiv 0 \bmod 4. \end{aligned} \tag{6.3}$$

Table 6.1: Exponents i, k of p in $N = a(\pi)$ and $a(\mathrm{sym}^3(\pi))$ respectively.

We use this table in Theorem 6.1.1 and Corollary 6.2.1. Here, $e = \frac{12}{(v_p(\Delta),12)}$ with v_p being the p-adic valuation.

Condition on $p \mid N$	Condition on E	i	k	Reduction type of E at p
for any p $p \mid\mid N$	$v_p(\Delta) > 0$, $v_p(c_4) = 0$	1	3	multiplicative reduction
for any p $p^i \mid\mid N$ $i \geq 2$	$j(E) \notin \mathbb{Z}_p$	$i = 2$ if $p \geq 3$ $i \in \{4, 6\}$ if $p = 2$	$2i$	additive, potential multiplicative
for $p \geq 5$ $p^2 \mid\mid N$	$j(E) \in \mathbb{Z}_p$, $e \in \{2, 4, 6\}$	2	4	additive, potential good
for $p \geq 5$ $p^2 \mid\mid N$	$j(E) \in \mathbb{Z}_p$, $e = 3$	2	2	additive, potential good
for $p = 3$ $p^2 \mid\mid N$	$j(E) \in \mathbb{Z}_3$, E satisfies P_2 or S_4 from Table 4.1	2	4	additive, potential good
for $p = 3$ $p^3 \mid\mid N$	$j(E) \in \mathbb{Z}_3$, E satisfies $\mathrm{S}_6^{'}$ from Table 4.1	3	5	additive, potential good
for $p = 3$ $p^4 \mid\mid N$	$j(E) \in \mathbb{Z}_3$, E satisfies P_3 or S_3 from Table 4.1	4	4	additive, potential good
for $p = 3$ $p^4 \mid\mid N$	$j(E) \in \mathbb{Z}_3$, E satisfies P_6 or S_6 from Table 4.1	4	6	additive, potential good
for $p = 3$ $p^5 \mid\mid N$	$j(E) \in \mathbb{Z}_3$, E satisfies $\mathrm{S}_6^{''}$ from Table 4.1	5	7	additive, potential good

To see this, note that $k = i$ in Table 6.1 precisely in the following cases:

1. $j(E) \in \mathbb{Z}_p$ and $e = 3$ for $p \geq 5$.

2. $j(E) \in \mathbb{Z}_3$ and E satisfies P_3 or S_3 from Table 4.1.

We have $e = \frac{12}{(v_p(\Delta),12)} = 3$ if and only if $v_p(\Delta) \equiv 0 \bmod 4$. Also, from Table 4.1, we see that E satisfies P_3 or S_3 if and only if $v_p(\Delta) \equiv 0 \bmod 4$. Hence, $k = i$ if and only if $v_p(\Delta) \equiv 0 \bmod 4$. Moreover, observe that $k = i + 2$ in Table 6.1 for all other cases except for the case when E has additive but potential multiplicative reduction at $p = 2$. Since we have discarded this case here, we get $k = i + 2$ if and only if $v_p(\Delta) \not\equiv 0 \bmod 4$. This proves (6.3) and hence proves (6.2). $\square$

6.2 Level of paramodular forms attached to elliptic curves over $\mathbb{Q}$

Since there is a natural correspondence between the cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ and paramodular forms as discussed in Section 3.5.4, we get the following results on paramodular forms coming from non-CM elliptic curves over $\mathbb{Q}$.

Corollary 6.2.1. *Let E be a non-CM elliptic curve over $\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$. Suppose that E has good or potential multiplicative reduction at $p = 2$. Then there is a cuspidal paramodular newform f of degree 2, weight 3, and level M such that $L(s, f) = L(s, E, \mathrm{sym}^3)$. Moreover, the conductor N of E and the level M of f are given by*

$$N = \prod_{p \mid \Delta} p^i \quad \textit{and} \quad M = \prod_{p \mid \Delta} p^k,$$

where the values of i, k are given in the Table 6.1 for p dividing the discriminant Δ of (2.71).

Proof. Note that, by Theorem 6.1.1, E corresponds to a cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ of weight 2 and conductor N such that $L(s, E, \mathrm{sym}^3) = L(s, \pi, \mathrm{sym}^3)$. Then, using Corollary 3.5.5, there is a scalar valued cuspidal paramodular newform f of degree 2, weight 3 such that $L(s, f) = L(s, \pi, \mathrm{sym}^3)$. So, we get $L(s, f) = L(s, E, \mathrm{sym}^3)$. Also, by Corollary 3.5.5, the level M of f is equal to the conductor $a(\mathrm{sym}^3(\pi))$ of $\mathrm{sym}^3(\pi)$. Hence, by (6.1) of Theorem 6.1.1, we get the value of N and M as stated above. $\square$

The next result is a special and more detailed version of Corollary 6.2.1. Here, we assume that the conductor N of E is given and E has good or multiplicative reduction at $p = 2$. We make such an assumption to get a compact formula for the paramodular level M.

Theorem 6.2.2. *Let E be a non-CM elliptic curve over $\mathbb{Q}$ given by the global minimal Weierstrass equation of the form (2.71) with coefficients in $\mathbb{Z}$ and conductor N. Let Δ be the discriminant attached to the given Weierstrass equation and v_p be the p-adic valuation. Let $\Delta' = 3^{-v_3(\Delta)}\Delta$. Suppose that E has good or multiplicative reduction at $p = 2$. Then there is a cuspidal paramodular newform f of degree 2, weight 3 and level M with the following properties:*

(i) The level M of f is given by $M = N \prod\limits_{\substack{p \mid N \\ v_p(\Delta) \not\equiv 0 \bmod 4}} p^2.$

(ii) The completed spin L-function $L(s, f)$ attached to f is given by

$$L(s, f) = \Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s + \frac{1}{2}\right)\prod_{p<\infty} L_p(s, f),$$

where $L_p(s,f) = L_p(s,E,\mathrm{sym}^3)$ *for all places* p. *If* $p|N$, *then*

$$L_p(s,f) = \begin{cases} \frac{1}{1-p^{-3/2-s}} & \text{if } E \text{ has split multiplicative reduction at } p, \\ \frac{1}{1+p^{-3/2-s}} & \text{if } E \text{ has non-split multiplicative reduction at } p, \\ \frac{1}{(1-\alpha p^{-s})(1-\alpha^{-1}p^{-s})} & \text{if } j(E) \in \mathbb{Z}_p \text{ and } v_p(\Delta) \equiv 0 \bmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Here, α *is an element of* $\mathbb{C}^\times$ *such that* $|\alpha| = 1$. *If the following condition is satisfied*

$$j(E) \in \mathbb{Z}_p, \ v_p(\Delta) \equiv 0 \bmod 4, \ \text{and} \ \left\{ \begin{array}{ll} (p-1)v_p(\Delta) \not\equiv 0 \bmod 12 & \text{if } p \geq 5, \\ \left(\frac{\Delta'}{3}\right) = -1 & \text{if } p = 3, \end{array} \right\},$$

then $\alpha = i$ *(the fourth root of unity).*

(iii) The Atkin-Lehner eigenvalues of f *at the finite places are given by*

$$\eta_p = \begin{cases} -1 & \text{if } p \mid N, \text{ and } E \text{ has split mult. red. at } p \text{ or satisfies } (6.4), \\ w(E/\mathbb{Q}_3) & \text{if } 3 \mid N, \ p = 3 \text{ and } E \text{ satisfies } \mathrm{S}_6' \text{ or } \mathrm{S}_6'', \\ 1 & \text{otherwise,} \end{cases}$$

where S_6', S_6'' *are defined in Table 4.1 and the condition* (6.4) *is given by*

$$j(E) \in \mathbb{Z}_p \ \text{and} \ \left\{ \begin{array}{ll} v_p(\Delta) \equiv 0 \bmod 4, (p-1)v_p(\Delta) \not\equiv 0 \bmod 12 & \text{if } p \geq 5, \\ v_p(\Delta) \equiv 2 \bmod 4, \left(\frac{\Delta'}{3}\right) = -1 & \text{if } p = 3, \end{array} \right\}. \quad (6.4)$$

Proof. Most parts of this theorem including the statement (i) follow from Corollary 6.1.2 and Corollary 6.2.1. Let $\pi \cong \otimes \pi_p$ be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ attached to $E/\mathbb{Q}$. By Corollary 6.2.1, we have $L_p(s,f) = L(s, \mathrm{sym}^3(\pi_p))$ for each prime $p \leq \infty$. Since f is a paramodular form of scalar weight 3, the archimedean component $\mathrm{sym}^3(\pi_\infty)$ of $\mathrm{sym}^3(\pi)$ is a holomorphic discrete series representation of weight $(3, 0)$. Then, using (2.49), we get

$$L_\infty(s, f) = L(s, \mathrm{sym}^3(\pi_\infty)) = \Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s + \frac{1}{2}\right). \tag{6.5}$$

Now the assertion (ii) follows from (6.5), Table 5.6 and Table 5.7. It is a well-known fact that the Atkin-Lehner eigenvalue η_p of f at a finite place p is equal to $\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_p)\right)$ (see Corollary 7.5.5 in [37]). Note from Table 5.9 that $\varepsilon\left(\frac{1}{2}, \mathrm{sym}^3(\pi_3)\right) = \varepsilon(\frac{1}{2}, \pi_3)$, when E satisfies S_6' or S_6''. Also, observe that $w(E/\mathbb{Q}_3) = \varepsilon(\frac{1}{2}, \pi_3)$, where the local root number $w(E/\mathbb{Q}_3)$ can be computed from Theorem 1.1. of [24] in terms of the Weierstrass coefficients of E. So, one can easily get the assertion (iii) using Table 5.8 and Table 5.9.

□

We have implemented some of our results in SAGE using data from LMFDB[1]. In Table 6.2 and Table 6.3, we give a list of examples to illustrate Theorem 6.2.2 using our SAGE program. To identify the local representations π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ attached to an elliptic curve $E/\mathbb{Q}$ at each prime $p \mid N$, we use the following result.

Remark 6.2.3. *Let* $\pi \cong \bigotimes_{p \leq \infty} \pi_p$ *be the cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ *attached to a non-CM elliptic curve* $E/\mathbb{Q}$*. For* $p \geq 5$*, the symbol* PS_e *(resp.* SC_e*) in Table 6.2 denotes the principal series representation* $\chi \times \chi^{-1}$ *(resp. dihedral supercuspidal representation* $\omega_{F,\xi}$*) of* $\mathrm{GL}(2, \mathbb{Q}_p)$ *in Theorem 4.2.1 (resp. Theorem 4.2.2) such that* $\chi|_{\mathfrak{o}_K^\times}$ *(resp.* $\xi|_{\mathfrak{o}_F^\times}$*) has order* $e \in \{2, 3, 4, 6\}$*. Similarly, for* $p = 3$*, the symbols* PS_i*,* SC_i*,* SC_6' *and* SC_6'' *in Table 6.3 denote the representation of* $\mathrm{GL}(2, \mathbb{Q}_3)$ *in Theorem 4.3.4 or Theorem 4.3.4 when* $E/\mathbb{Q}_3$ *satisfies* P_i*,* S_i*,* S_6'*,* S_6'' *respectively for* $i \in \{3, 4, 6\}$*. Also,* St*,* $\mathrm{St}_{\mathrm{rm}}$*,* $\mathrm{St}_{\mathrm{ur}}$ *denote the Steinberg representation, a ramified twist of Steinberg representation, an unramified twist of Steinberg representation of* $\mathrm{GL}(2, \mathbb{Q}_p)$ *respectively.*

[1] LMFDB - The L-functions and Modular Forms Database

Table 6.2: Examples of paramodular forms coming from elliptic curves when $3^2 \nmid N$.

Here $\pi \cong \otimes_p \pi_p$ is the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ associated to a non-CM $E/\mathbb{Q}$ of conductor N and f is the paramodular newform of level M attached to E by Theorem 6.2.2. Also, $\Pi_p = \mathrm{sym}^3(\pi_p)$ are given by representation types, $\epsilon_p = \varepsilon(1/2, \pi_p)$ and $\eta_p = \varepsilon(1/2, \Pi_p)$. SC_e, PS_e, St, $\mathrm{St}_{\mathrm{rm}}$ are defined in Remark 6.2.3 and α is as in Theorem 6.2.2.

W.E. of $E/\mathbb{Q}$	N	M	p	$L_p(s,E)$	ϵ_p	π_p	$L_p(s,f)$	η_p	Π_p
$y^2+xy=$ x^3+x^2-2x-7	11^2	11^2	11	1	-1	SC_3	$\frac{1}{1+11^{-2s}}$	-1	X
$y^2+xy+y=$ $x^3+x^2-30x-76$	11^2	11^4	11	1	-1	SC_6	1	1	X
$y^2+y=$ $x^3-x^2-40x-221$	11^2	11^4	11	1	-1	$\mathrm{St}_{\mathrm{rm}}$	1	1	IVa
$y^2+y=$ $x^3-7x+12$	$5\cdot 7^2$	$5^3\cdot 7^4$	5	$\frac{1}{1-5^{-1/2-s}}$	-1	St	$\frac{1}{1-5^{-3/2-s}}$	-1	IVa
			7	1	-1	SC_4	1	1	VIII
$y^2+y=$ $x^3+x^2-8x+19$	$5^2\cdot 11$	$5^4\cdot 11^3$	5	1	1	PS_2	1	1	I
			11	$\frac{1}{1-11^{-1/2-s}}$	-1	St	$\frac{1}{1-11^{-3/2-s}}$	-1	IVa
$y^2+xy=$ $x^3-393x-3298$	$5^2\cdot 7^2$	$5^4\cdot 7^2$	5	1	-1	PS_4	1	1	I
			7	1	1	PS_3	$\frac{1}{(1-\alpha 7^{-s})(1-\alpha^{-1}7^{-s})}$	1	I
$y^2+xy+y=$ x^3+x^2-8x+6	$5^2\cdot 7^2$	$5^4\cdot 7^4$	5	1	-1	PS_4	1	1	I
			7	1	-1	PS_6	1	1	I

Table 6.3: Examples of paramodular forms coming from elliptic curves when $3^2 \mid N$.

Here $\pi \cong \otimes_p \pi_p$ is the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ associated to a non-CM $E/\mathbb{Q}$ of conductor N and f is the paramodular newform of level M attached to E by Theorem 6.2.2. Also, $\Pi_p = \mathrm{sym}^3(\pi_p)$ are given by representation types, $\epsilon_p = \varepsilon(1/2, \pi_p)$ and $\eta_p = \varepsilon(1/2, \Pi_p)$. SC_e, PS_e, St, $\mathrm{St}_{\mathrm{ra}}$ are defined in Remark 6.2.3 and α is as in Theorem 6.2.2.

W.E. of $E/\mathbb{Q}$	N	M	p	$L_p(s,E)$	ϵ_p	π_p	$L_p(s,f)$	η_p	Π_p
$y^2+xy=x^3-x^2+6x$	$2\cdot 3^2\cdot 5$	$2^3\cdot 3^4\cdot 5^3$	2	$\frac{1}{1+2^{-1/2-s}}$	1	$\mathrm{St}_{\mathrm{ur}}$	$\frac{1}{1+2^{-3/2-s}}$	1	IVa
			3	1	1	SC_4	1	1	VIII
			5	$\frac{1}{1-5^{-1/2-s}}$	-1	St	$\frac{1}{1-5^{-3/2-s}}$	-1	IVa
$y^2+xy+y=x^3-x^2+4x-1$	$2\cdot 3^4$	$2^3\cdot 3^4$	2	$\frac{1}{1-2^{-1/2-s}}$	-1	St	$\frac{1}{1-2^{-3/2-s}}$	-1	IVa
			3	1	1	SC_3	$\frac{1}{1+3^{-2s}}$	1	X
$y^2+xy=x^3-x^2-6x+8$	$2\cdot 3^4$	$2^3\cdot 3^6$	2	$\frac{1}{1+2^{-1/2-s}}$	1	$\mathrm{St}_{\mathrm{ur}}$	$\frac{1}{1+2^{-3/2-s}}$	1	IVa
			3	1	1	SC_6	1	-1	X
$y^2+xy+y=x^3-x^2-5x+5$	$2\cdot 3^4$	$2^3\cdot 3^4$	2	$\frac{1}{1-2^{-1/2-s}}$	-1	St	$\frac{1}{1-2^{-3/2-s}}$	-1	IVa
			3	1	1	PS_3	$\frac{1}{(1-\alpha 3^{-s})(1-\alpha^{-1}3^{-s})}$	1	I
$y^2+xy=x^3-x^2+3x-1$	$2\cdot 3^4$	$2^3\cdot 3^6$	2	$\frac{1}{1+2^{-1/2-s}}$	1	$\mathrm{St}_{\mathrm{ur}}$	$\frac{1}{1+2^{-3/2-s}}$	1	IVa
			3	1	-1	PS_6	1	1	I
$y^2+xy=x^3-x^2-3x+3$	$2\cdot 3^3$	$2^3\cdot 3^5$	2	$\frac{1}{1+2^{-1/2-s}}$	1	$\mathrm{St}_{\mathrm{ur}}$	$\frac{1}{1+2^{-3/2-s}}$	1	IVa
			3	1	-1	SC_6'	1	-1	SC
$y^2+xy=x^3-x^2+3x+5$	$2\cdot 3^5$	$2^3\cdot 3^7$	2	$\frac{1}{1+2^{-1/2-s}}$	1	$\mathrm{St}_{\mathrm{ur}}$	$\frac{1}{1+2^{-3/2-s}}$	1	IVa
			3	1	1	SC_6''	1	1	SC

Index

Bibliography

[1] Andrianov, A. N., and Kalinin, V. L. On the analytic properties of standard zeta functions of siegel modular forms. *Math. USSR, Sb. 35* (1979), 1–17.

[2] Arthur, J. *The endoscopic classification of representations*, vol. 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[3] Asgari, M., and Schmidt, R. Siegel modular forms and representations. *Manuscripta Math. 104*, 2 (2001), 173–200.

[4] Borel, A. Automorphic L-functions. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[5] Borel, A., and Jacquet, H. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 189–207. With a supplement "On the notion of an automorphic representation" by R. P. Langlands.

[6] Brumer, A., and Kramer, K. Paramodular abelian varieties of odd conductor. *Trans. Amer. Math. Soc. 366*, 5 (2014), 2463–2516.

[7] Bump, D. *Automorphic forms and representations*, vol. 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[8] Bump, D., Cogdell, J. W., de Shalit, E., Gaitsgory, D., Kowalski, E., and Kudla, S. S. *An introduction to the Langlands program*. Birkhäuser Boston, Inc., Boston, MA, 2003. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001, Edited by Joseph Bernstein and Stephen Gelbart.

[9] Bushnell, C. J., and Henniart, G. *The local Langlands conjecture for* $\mathrm{GL}(2)$, vol. 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[10] Diamond, F., and Shurman, J. *A first course in modular forms*, vol. 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[11] Fröhlich, A., and Queyrut, J. On the functional equation of the Artin L-function for characters of real representations. *Invent. Math. 20* (1973), 125–138.

[12] Gan, W. T., and Takeda, S. The local Langlands conjecture for GSp(4). *Ann. of Math. (2) 173*, 3 (2011), 1841–1882.

[13] Gelbart, S., and Jacquet, H. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4) 11*, 4 (1978), 471–542.

[14] Gelbart, S. S. *Automorphic forms on adèle groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975. Annals of Mathematics Studies, No. 83.

[15] Harris, M., and Taylor, R. *The geometry and cohomology of some simple Shimura varieties*, vol. 151 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[16] Henniart, G. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. *Invent. Math. 139*, 2 (2000), 439–455.

[17] Kim, H. H., and Shahidi, F. Symmetric cube L-functions for GL_2 are entire. *Ann. of Math. (2) 150*, 2 (1999), 645–662.

[18] Kim, H. H., and Shahidi, F. Cuspidality of symmetric powers with applications. *Duke Math. J. 112*, 1 (2002), 177–197.

[19] Kim, H. H., and Shahidi, F. Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. *Ann. of Math. (2) 155*, 3 (2002), 837–893. With an appendix by Colin J. Bushnell and Guy Henniart.

[20] Klingen, H. *Introductory lectures on Siegel modular forms*, vol. 20 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[21] Knapp, A. W. *Representation theory of semisimple groups*, vol. 36 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[22] Knapp, A. W. Local Langlands correspondence: the Archimedean case. In *Motives (Seattle, WA, 1991)*, vol. 55 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994, pp. 393–410.

[23] Knapp, A. W. Introduction to the Langlands program. In *Representation theory and automorphic forms (Edinburgh, 1996)*, vol. 61 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1997, pp. 245–302.

[24] Kobayashi, S.-i. The local root number of elliptic curves with wild ramification. *Math. Ann. 323*, 3 (2002), 609–623.

[25] Koblitz, N. *Introduction to elliptic curves and modular forms*, second ed., vol. 97 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1993.

[26] Kraus, A. Courbes elliptiques semi-stables sur les corps de nombres. *Int. J. Number Theory 3*, 4 (2007), 611–633.

[27] Kudla, S. S. The local Langlands correspondence: the non-Archimedean case. In *Motives (Seattle, WA, 1991)*, vol. 55 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994, pp. 365–391.

[28] Lansky, J., and Raghuram, A. On the correspondence of representations between GL(n) and division algebras. *Proc. Amer. Math. Soc. 131*, 5 (2003), 1641–1648.

[29] Loeffler, D., and Weinstein, J. On the computation of local components of a newform. *Math. Comp. 81*, 278 (2012), 1179–1200.

[30] Moy, A., and Prasad, G. Unrefined minimal K-types for p-adic groups. *Invent. Math. 116*, 1-3 (1994), 393–408.

[31] Moy, A., and Prasad, G. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv. 71*, 1 (1996), 98–121.

[32] Neukirch, J. *Algebraic number theory*, vol. 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[33] Papadopoulos, I. Sur la classification de Néron des courbes elliptiques en caractéristique résiduelle 2 et 3. *J. Number Theory 44*, 2 (1993), 119–152.

[34] Poor, C., and Yuen, D. S. Paramodular cusp forms. *Math. Comp. 84*, 293 (2015), 1401–1438.

[35] Ramakrishnan, D., and Shahidi, F. Siegel modular forms of genus 2 attached to elliptic curves. *Math. Res. Lett. 14*, 2 (2007), 315–332.

[36] Roberts, B., and Schmidt, R. On modular forms for the paramodular groups. In *Automorphic forms and zeta functions*. World Sci. Publ., Hackensack, NJ, 2006, pp. 334–364.

[37] Roberts, B., and Schmidt, R. *Local newforms for GSp(4)*, vol. 1918 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[38] Rohrlich, D. E. Variation of the root number in families of elliptic curves. *Compositio Math. 87*, 2 (1993), 119–151.

[39] Rohrlich, D. E. Elliptic curves and the Weil-Deligne group. In *Elliptic curves and related topics*, vol. 4 of *CRM Proc. Lecture Notes*. Amer. Math. Soc., Providence, RI, 1994, pp. 125–157.

[40] Sally, Jr., P. J., and Tadić, M. Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$. *Mém. Soc. Math. France (N.S.)*, 52 (1993), 75–133.

[41] Schmidt, R. Some remarks on local newforms for GL(2). *J. Ramanujan Math. Soc. 17*, 2 (2002), 115–147.

[42] Schmidt, R. Archimedean aspects of Siegel modular forms of degree 2. *Rocky Mountain J. Math. 47*, 7 (2017), 2381–2422.

[43] Schmidt, R. Packet structure and paramodular forms. *Trans. Amer. Math. Soc. 370*, 5 (2018), 3085–3112.

[44] Schmidt, R. Packet structure and paramodular forms. *Trans. Amer. Math. Soc. 370*, 5 (2018), 3085–3112.

[45] Schmidt, R., and Turki, S. Triply imprimitive representations of GL(2). *Proc. Amer. Math. Soc. 146*, 3 (2018), 971–981.

[46] Serre, J.-P. *Local fields*, vol. 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[47] Silverman, J. H. *Advanced topics in the arithmetic of elliptic curves*, vol. 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[48] Silverman, J. H. *The arithmetic of elliptic curves*, second ed., vol. 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[49] Tate, J. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26.

[50] Turki, S. *The representations of p-adic fields associated to elliptic curves.* PhD thesis, University of Oklahoma, 2015.